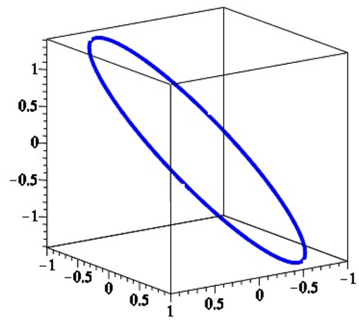
MATH 223

Some Notes on Assignment 34

Exercise 43 of Chapter 8.

Sketch the curve γ parameterized by $g(t) = (\sin t, \cos t, \sin t - \cos t), 0 \le t \le 2\pi$. Verify Stokes' Theorem for γ and the vector field $\mathbf{F}(x, y, z) = (yz, xz, xy)$.



We need to show $\int_S$ curl $\mathbf{F} \cdot dS = \int_{\partial S} \mathbf{F} \cdot \mathbf{x}$.

The curl of the vector field $\mathbf{F}(x, y, z) = (yz, xz, xy)$ is

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{pmatrix} = [(xy)_y - (xz)z]\,\mathbf{i} - [(xy)_x - (yz)_z[\,\mathbf{j} + [(xz)_x - (yz)y]\,\mathbf{k}$$

$$= (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k} = (0, 0, 0)$$

Hence $\int_S$ curl $\mathbf{F} \cdot dS = 0$.

Now $\mathbf{g}(t) = (\sin t, \cos t, \sin t - \cos t)$ gives $\mathbf{g}'(t) = (\cos t, -\sin t, \cos t + \sin t)$ and

$$\mathbf{F}(\mathbf{g}(t)) = (\sin t \cos t - \cos^2 t, \sin^2 t - \sin t \cos t, \sin t \cos t).$$

After a bit of algebra, we find

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = 2 \sin t \cos^2 t + 2 \sin^2 t \cos t - \left[\sin^3 t + \cos^3 t\right]$$

Now the line integral of $\mathbf{F}$ around the boundary of S is the line integral over the curve γ with parametrization $\mathbf{g}$; that is,

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{x} = \int_{t=0}^{t=2\pi} 2 \sin t \cos^2 t + 2 \sin^2 t \cos t - \sin^3 t - \cos^3 t \, dt$$

but each of the four terms in the integrand $2 \sin t \cos^2 t, 2 \sin^2 t \cos t, -\sin^3 t, -\cos^3 t)$ has a definite integral equal to 0 over the interval $[0, 2\pi]$ and hence the line integral also has value 0.

Note:

$$\int \sin t \cos^2 t \, dt = \frac{-\cos^3 t}{3}, \int \sin^2 \cos t \, dt = \frac{\sin^3 t}{3}$$

$$\sin^3 t = \sin^2 t \sin t = (1 - \cos^2 t) \sin t = \sin t - \sin t \cos^2 \text{ gives } \int \sin^3 t \, dt = \frac{\cos^3 t}{3} - \cos t$$

A much simpler way to determine the line integral is to observe that F is a conservative vector field with potential function *f(x,y,z) =xyz* .