MATH 223

Some Notes on Assignment 33

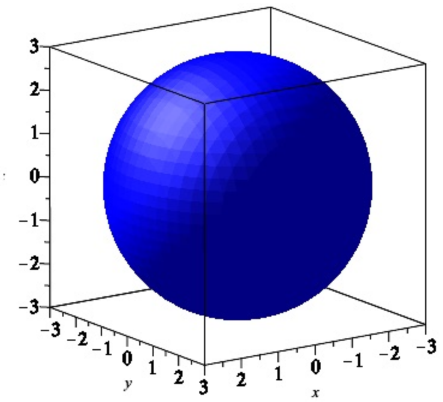
Exercises 36ac, 38, and 39 of Chapter 8.

36: Sketch a picture of each of the regions $\mathcal{R}$ in $\mathbb{R}^3$ described below along with a representative number of outward-pointing normals.Then verify the correctness of Gauss's Theorem for the given vector fields.

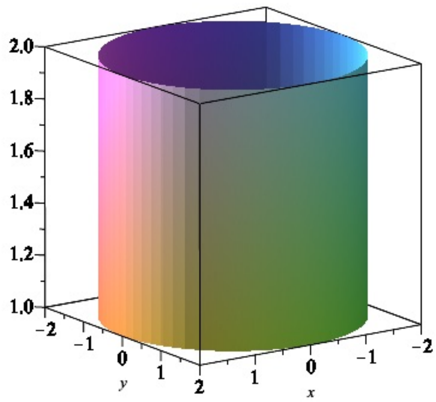
(a): $\mathcal{R} : x^2 + y^2 + z^2 \leq 9; \mathbf{F}(x, y, z) = (y, -x, 0)$

(c): $\mathcal{R} : x^2 + y^2 \leq 4, 1 \leq z \leq 2; \mathbf{F}(x, y, z) = (0, y, 0)$

Solution: We need to evaluate $\int_{\mathcal{R}}$ div $\mathbf{F}$ and $\int_{\partial\mathcal{R}} \mathbf{F}$ independently of each other and show they are equal.



Exercise 36a



Exercise 36c

(a) See Solution to Exercise 27 of Assignment 33. The divergence of $\mathbf{F}$ is $y_x + (-x)_y + 0_z = 0$ so $\int_{\mathcal{R}}$ div $\mathbf{F} = 0$. Using the parametrization from Exercise 27, $\sigma(s, t) = (3 \cos s \sin t, 3 \sin s \sin t, 3 \cos t)$, we have $\sigma_s(s, t) \times \sigma_t(s, t) = \left(-9 \cos s \sin^2 t, -9 \sin s \sin^2 t, -9 \sin t \cos t\right)$. We also have $F(\sigma(s, t)) = (3 \sin s \sin t, -3 \cos s \sin t, 0)$. Then $F(\sigma(s, t)) \cdot (\sigma_s(s, t) \times \sigma_t(s, t)) = -27 \sin s \cos s \sin^3 t + 27 \sin s \cos s \sin^3 t = 0$. Thus $\int_{\partial\mathcal{R}} \mathbf{F} = \int_{\partial\mathcal{R}} 0 = 0$.

(c) div $\mathbf{F} = 0_x + y_y + 0_z = 1$ so $\int_{\mathcal{R}}$ div $\mathbf{F} = \int_{\mathcal{R}} 1 =$ Volume enclosed by $\mathcal{R}$ but this is the volume of a circular cylinder of radius 2 and height 1 which has volume 4π.

The base of the cylinder lies in the xy-plane. The surface has three components:

The Top T:	The Bottom B:	The Cylindrical Side C: $(x^2 + y^2 = 4, 1 \leq z \leq 2)$
$(x^2 + y^2 \leq 4z = 2)$	$(x^2 + y^2 \leq 4, z = 1)$	$(x^2 + y^2 = 4, 1 \leq z \leq 2)$

We can parametrize them as follows:
Top T: $f(s, t) = (2s \cos t, 2s \sin t, 2), 0 \leq s \leq 1, 0 \leq t \leq 2\pi$
Side C : $g(s, t) = (2 \cos s, 2 \sin s, t), 0 \leq s \leq 2\pi, 0 \leq t \leq 1)$
Bottom B: $h(s, t) = (2t \cos s, 2t \sin s, 1), 0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi$

Then

$f_s(s, t) = (2 \cos t, 2 \sin t, 0)$	$f_t(s, t) = (-2s \sin t, 2s \cos t, 0)$	$f_s \times f_t = (0, 0, 4s)$ points straight up
$g_s(s, t) = (-2 \sin s, 2 \cos s, 0)$	$g_t(s, t) = (0, 0, 1)$	$g_s \times g_t = (2 \cos s, 2 \sin s, 0)$ points outward
$h_s(s, t) = (-2t \sin s, 2t \cos s, 0)$	$h_t(s, t) = (2 \cos s, 2 \sin s, 0)$	$h_s \times h_t = (0, 0, -4t$ points straight down

We also have $\mathbf{F}(f(s, t)) = (0, 2s \sin t, 0)$ for the top T, $\mathbf{F}(g(s, t)) = (0, 2 \sin s, 0)$ for the side C and $\mathbf{F}(h(s, t)) = (0, 2t \cos s, 0)$ for the bottom B. This yields $\mathbf{F}(\mathbf{f}) \cdot (f_s \times f_t) = 0, \mathbf{F}(g) \cdot (g_s \times g_t) = 4 \sin^2 s$, and $\mathbf{F}(h) \cdot (h_s \times h_t) = 0$.

Thus $\int_{\partial\mathcal{R}} \mathbf{F} = \int_B \mathbf{F} + \int_T \mathbf{F} + \int_C \mathbf{F} = 0 + 0 + \int_C \mathbf{F} = \int_C \mathbf{F} = \int_{t=0}^{t=1} \int_{s=0}^{s=2\pi} 4\sin^2 s\, ds\, dt$.

Finally,

$$\int_{t=0}^{t=1} \int_{s=0}^{s=2\pi} 4\sin^2 s ds dt = \int_{t=0}^{t=1} \int_{s=0}^{s=2\pi} 2(1-\cos 2s) ds dt = \int_{t=0}^{t=1} [2s - sin2S]_{s=0}^{s=2\pi} dt = \int_{t=0}^{t=1} 4\pi dt = 4\pi.$$

38: Find the flux out of the upper hemisphere $\mathcal{H}$ of the unit sphere in $\mathbb{R}^3$ of the vector field $\mathbf{F}(x, y, z) = (x + y, y + z, x + z)$.

Solution: div $\mathbf{F} = (x+y)_x + (y+z)_y + (x+z)_z = 1+1+1 = 3$ so the flux is 3 times the volume enclosed by the upper hemisphere (see Exercise 39); that is, $3\left(\frac{4}{3}\right)\pi 1^3 = 4\pi$.

39: Show that the volume of a region $\mathcal{R}$ in $\mathbb{R}^3$ is equal to

$$\frac{1}{3}\int_{\partial R} \mathbf{F} \text{ if } \mathbf{F}(x, y, z) = (x, y, z).$$

Solution: div $\mathbf{F} = x_x + y_y + z_z = 1 + 1 + 1 = 3$. By Gauss,

$$\frac{1}{3}\int_{\partial\mathcal{R}} \mathbf{F} = \frac{1}{3}\int_{\mathcal{R}} \text{div } \mathbf{F} = \frac{1}{3}\int_{\mathcal{R}} 3 = \int_{\mathcal{R}} 1 = \text{Volume of } \mathcal{R}$$