MATH 223

Some Notes on Assignment 31

Exercises 15, 17, 20 21ac, 23 and 26 of Chapter 8.

15: Find a function f so that $\nabla f(x,y) = (ye^x, e^x)$ or show no such f exists.

Solution: The vector field $F(x,y), G(x,y) = (ye^x, e^x)$ has $G_x = e^x = F_y$ so it is possible that it is a gradient field. To begin to construct a potential function, observe that $\int ye^x \, dx = ye^x + h(y)$ for some function h of y. Taking the derivative of this expression with respect to y, we have $e^x + h'(y)$ which matches $G(x,y)$ if we take any constant function for h. Thus a potential function if $f(x,y) = ye^x$

17: Find a function f so that $\nabla f(x,y) = (4xe^{2x^2+3y}, 4e^{2x^2+3y}))$ or show no such f exists.

Solution: The vector field $F(x,y), G(x,y) = (4xe^{2x^2+3y}, 4e^{2x^2+3y})$ has $G_x = 8xe^{2x^2+3y}$ but $F_y = 12xe^{2x^2+3y}$. Thus $(4xe^{2x^2+3y}, 4e^{2x^2+3y})$ is not a gradient field; there is no potential function f.

20: Find a function f so that $\nabla f(x,y,z) = (yz, xz, -zy)$ or show no such f exists.

Solution: If there such a function f, then its Jacobian matrix would be symmetric but the Jacobian is

$$\begin{pmatrix} 0 & z & y \\ z & 0 & x \\ 0 & -z & -y \end{pmatrix}.$$

which is not symmetric. In particular, although $f_{xy} = z = f_{yx}$, we have $f_{xz} = y \neq 0 = f_{zx}$ and $f_{yz} = x \neq -z = f_{zy}$. Thus no such function f exists.

21ac: Use Green's Theorem to compute the line integral of the vector field $\mathbf{F}(x,y) = (3y, 2x^2)$ around each of the curves γ described below:

(a) The circle γ described by $\mathbf{g}(t) = (2\cos t, 2\sin t), 0 \leq t \leq 2\pi$.
Solution: $\mathbf{g}$ traces out the circle of radius 2 centered at the origin in a counterclockwise direction. The curve bounds the disk of radius 2 centered at the origin. By Green's Theorem

$$\int_\gamma \mathbf{F} = \int_\gamma (3y, 2x^2) = \int_\mathcal{R} (2x^2)_x - (3y)_y \, dx\, dy = \int_\mathcal{R} 4x - 3 \, dy\, dx$$

We can evaluate this multiple integral as

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=-\sqrt{4-x^2}} (4x-3)\, dy\, dx \text{ or switch to polar coordinates } \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (8\cos\theta - 3)\, r\, dr\, d\theta :$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (8\cos\theta - 3)\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} (8\cos\theta - 3) \left[\frac{r^2}{2}\right]_{r=0}^{r=2} d\theta$$
$$= \int_{\theta=0}^{\theta=2\pi} (16\cos\theta - 6)\, d\theta = [16\sin\theta - 6\theta]_{\theta=0}^{\theta=2\pi} = -12\pi$$

(c) The square with vertices (0,0), (1,0), (1,1), (0,1) traced counterclockwise.
Solution: Green's Theorem, the line integral has value

$$\int_{y=0}^{y=1} \int_{x=0}^{x=1} (4x-3)\, dx\, dy = \int_{y=0}^{y=1} \left[2x^2 - 3x\right]_{x=0}^{x=1} dy = \int_{y=0}^{y=1} -1\, dy = -1.$$

23: Let the vector field $\mathbf{F}$ be defined by $\mathbf{F}(x, y) = (-y, x)$ and $\mathcal{R}$ a simple region with a smooth boundary curve γ traced counterclockwise. If A is the area of the region, show $2A = \int_\gamma \mathbf{F}$.

Solution: By Green"s Theorem,

$$\int_\gamma \mathbf{F} = \int_\mathcal{R} \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\, dx\, dy = \int_\mathcal{R} 1 - (-1)\, dx\, dy = \int_\mathcal{R} 2\, dx\, dy = 2 \text{ Area of } \mathcal{R}$$

26: The **Laplacian** Δf of a twice continuously differentiable function f is equal to $f_{xx} + f_{yy}$. If $\mathbf{F} = \nabla f$, show that Green's Theorem takes the form

$$\int_\gamma \nabla f \cdot \mathbf{n} = \int_D \Delta f$$

Solution:

$$\int_\gamma \nabla f \cdot \mathbf{n} = \int_\gamma \mathbf{F} \cdot \mathbf{n} = \int_\gamma \operatorname{div} \mathbf{F} = \int_D \operatorname{div}(f_x, f_y) = \int_D f_{xx} + f_{yy} = \int_D \Delta f$$