MATH 223

Some Notes on Assignment 29

Exercises 23 and 30 of Chapter 7.

23 Find the surface area of a circular cylinder of radius r and height h by rotating the graph of $f(x) = r, 0 \le x \le h$ about the x-axis.

Solution: Using the parametrization $g(t) = (t, r), 0 \le t \le h$, we have $g'(t) = (1, 0)$ so $|g'(t)| = 1$ and surface area is $\int_0^h 2\pi 1\, dt = 2\pi h$.

30: Sketch the solid obtained by revolving the graph of $y = 4\sqrt[3]{x}$ from (8,8) to (27, 12) around the $y-$axis and determine its surface area.
Solution: Let $g(t) = (t, 4\sqrt[3]{t}), 8 \le t \le 27$ be the parametrization. Then $g'(t) = \left(1, \frac{4}{3}t^{-2/3}\right) = (1, \frac{4}{3t^{2/3}})$ so

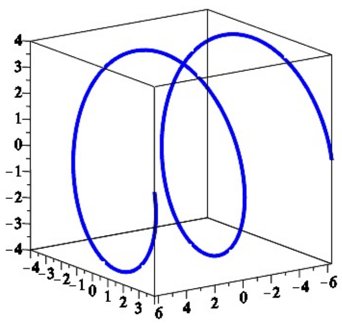
$$|g'(t)| = \sqrt{1 + \frac{16}{9t^{4/3}}} = \frac{\sqrt{9t^{4/3} + 16}}{3t^{2/3}}$$

Then the surface area obtained by revolving about the **y** axis is

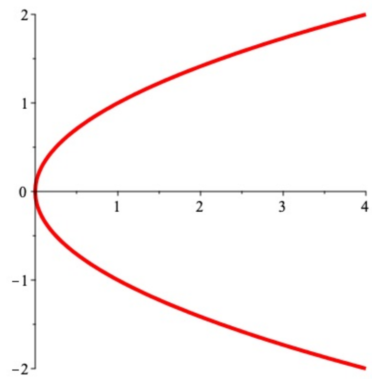
$$\begin{aligned}\int_8^{27} 2\pi t \frac{\sqrt{9t^{4/3}+16}}{3t^{2/3}}\, dt &= 2\pi \int_8^{27} t^{1/3}\sqrt{9t^{4/3}+16}\, dt \\ &= 2\pi \frac{1}{54}\left[(9t^{4/3}+16)^{3/2}\right]_8^{27} \\ &= \frac{\pi}{27}\left(745^{3/2} - 160^{3/2}\right)\end{aligned}$$

Exercise A: A curve γ has the parametrization $g(t) = (t, 4\cos t, 4\sin t)$ Sketch the curve, find its curvature and show it is constant.

Solution: We have $g'(t) = (1, -4\sin t, 4\cos t)$ so $|g'(t)| = \sqrt{1 + 16\sin^2 t + 16\cos^t} = \sqrt{1+16} = \sqrt{17}$. Thus the unit tangent vector is $\mathbf{T} = \frac{1}{\sqrt{17}}(1, -4\sin t, 4\cos t)$ and $\mathbf{T}' = \frac{1}{\sqrt{17}}(0, -4\cos t, -4\sin t$ so $|\mathbf{T}'| = \frac{1}{\sqrt{17}}\sqrt{0 + 16\cos^2 + 16\sin^2 t} = \frac{4}{\sqrt{17}}$. This curvature is $\kappa = \frac{|\mathbf{T}'|}{|g'|} = \frac{4}{\sqrt{17}}\frac{1}{\sqrt{17}} = \frac{4}{17}$.



Graph of $g(t) = (t, 4\cos t, 4\sin t), -2\pi \le t \le 2\pi$



Graph of $g(t) = (t^2, t), -2 \le t \le 2$

Exercise B: Sketch the curve with parametrization $g(t) = (t^2, t), -2 \le t \le 2$ and find its curvature at $t = 0$ and at $t = \sqrt{6}$.

Solution: . We have $g'(t) = 2t, 1), |g'(t)| = \sqrt{1+4t^2}$ so

$$\mathbf{T}(t) = \left(\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right) \text{ with } \mathbf{T}'(t) = \left(\frac{2}{\sqrt{1+4t^2}^{3/2}}, \frac{-4t}{\sqrt{1+4t^2}^{3/2}}\right) \text{ and } |\mathbf{T}'(t)| = \frac{2}{1+4t^2}$$

$$\text{Thus } \kappa(t) = \frac{2}{1+4t^2}\frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}$$

which makes $\kappa(0) = 2$ and $\kappa(\sqrt{6}) = \frac{2}{(1+24)^{3/2}} = \frac{2}{5^3}$.

Exercise C: Suppose the curve C in the plane is the graph of the real-valued function $y = f(x)$ of one variable. Show that its curvature is

$$\frac{|f''(x)|}{(1+|f'(x)|^2)^{3/2}}$$

Solution: To simplify the notation, we'll use F for the first derivative f' and S for the second derivative f'', simply writing f for $f(x)$, F for $f'(x)$, and S for $f''(x)$.

Then the parametrization $g(x) = (x, f(x))$ has $g' = (1, F)$ so $|g'| = \sqrt{1+F^2}$. Then

$$\mathbf{T} = \left(\frac{1}{\sqrt{1+F^2}}, \frac{F}{\sqrt{1+F^2}}\right) \text{ and } \mathbf{T}' = \left(\frac{-FS}{(1+F^2)^{3/2}}, \frac{S}{(1+F^2)^{3/2}}\right)$$

(leaving out some intermediate steps in calculating $\mathbf{T}'$) which makes

$$|\mathbf{T}'| = \sqrt{\frac{F^2S^2+S^2}{(1+F^2)^3}} = \sqrt{\frac{S^2(1+F^2)}{(1+F^2)^3}} = \sqrt{\frac{S^2}{(1+F^2)^2}} = \frac{|S|}{1+F^2}$$

$$\text{Thus } \kappa = \frac{|\mathbf{T}'|}{|g'|} = \frac{|S|}{1+F^2}\frac{1}{\sqrt{1+F^2}} = \frac{|S|}{(1+F^2)^{3/2}} = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}$$

Exercise D: If C is a curve in 3-dimensional space with parametrization $g(t)$, show that its curvature is given by

$$\frac{|g'(t)\times g''(t)|}{|g'(t)|^3}$$

.

Solution: Note first that $|\mathbf{T}| = 1$ so $\mathbf{T}\cdot\mathbf{T} = |\mathbf{T}|^2 = 1$. Taking the derivative of both sides with respect to t, we have $\mathbf{T}\cdot\mathbf{T}' + \mathbf{T}'\cdot\mathbf{T} = 0$ or $2\mathbf{T}\cdot\mathbf{T}' = 0$. Hence $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal to each other.

Next note that $\mathbf{T} = \frac{g'}{|g'|}$ so $g' = |g'|\mathbf{T}$. To get g'' into the picture, differentiate this last equation with respect to t using the Product Rule:

$$g'' = |g'|\mathbf{T}' + |g'|'\mathbf{T}. \text{ Note that } |g'| \text{ and } |g'|' \text{ are scalars.}$$

Then

$$g' \times g" = g' \times (|g'|\mathbf{T}' + |g'|'\mathbf{T}) = g' \times |g'|\mathbf{T}' + g' \times |g'|'\mathbf{T}$$

Now use $g' = |g'|\mathbf{T}$ and that $|g'|$ and $|g'|'$ are scalars to write

$$g' \times g" = |g'|\mathbf{T} \times |g'|\mathbf{T}' + |g'|\mathbf{T} \times |g'|'\mathbf{T} = |g'||g'|\mathbf{T}\times\mathbf{T}' + |g'||g'|'\mathbf{T}\times\mathbf{T}$$

Now $\mathbf{T}\times\mathbf{T}$ is zero $\mathbf{T}$ is parallel to itself so

$$g' \times g" = |g'||g'|(\mathbf{T}\times\mathbf{T}') \text{ so } |g' \times g"| = |g'|^2|\mathbf{T}\times\mathbf{T}'|$$

But $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal so the angle θ between them is $\pi/2$. Thus

$$|g' \times g"| = |g'|^2|\mathbf{T}||\mathbf{T}'|\sin\pi/2 = |g'|^2|\mathbf{T}||\mathbf{T}'|1 = |g'|^2|\mathbf{T}'| \text{ since } |\mathbf{T}| = 1$$

Dividing through by $|g'|^3$ gives

$$\frac{|g'(t)\times g''(t)|}{|g'(t)|^3} = \frac{|\mathbf{T}'|}{|g'|} = \kappa$$