MATH 223

Some Notes on Assignment 26

Exercises 2, 4ad, 5, 7 and 8 of Chapter 7.

2: Nigel must apply a force of 400 Newtons to push his lorry 10 meters along a level street. Find the work done.
Solution: Work $= 400 \times 10 = 4000$ newton-meters.

4: Find the work done by the planar vector field $\mathbf{F}(x, y) = (3x + 2y, 2x + 3y)$ along each of following curves:

(a) : $y = x^3$ from (0,0) to (1,1)
Solution: Let $g(t) = (t, t^3), 0 \leq t \leq 1$ parametrize the curve. Then $g'(t) = (1, 3t^2)$ and $F(g(t)) = F(t, t^3) = \left(3t + 2t^3, 2t + 3t^3\right)$, yielding $F(g(t)) \cdot g'(t) = \left(3t + 2t^3, 2t + 3t^3\right) \cdot (1, 3t^2) = 3t + 2t^3 + 6t^3 + 9t^4 = 3t + 8t^3 + 9t^5$
Thus the work done $= \int_0^1 3t + 8t^3 + 9t^5 \, dt = \left[\frac{3}{2}t^2 + 2t^4 + \frac{3}{2}t^6\right]_{t=0}^{t=1} = 5$

(d): $\mathbf{g}(t) = (t^2, t^3), 1 \leq t \leq 4$
Solution: Here $g'(t) = (2t, 3t^2)$ and $F(g(t)) = F(t, t^3) = \left(3t^2 + 2t^3, 2t^2 + 3t^3\right)$, making $F(g(t)) \cdot g'(t) = \left(3t^2 + 2t^3, 2t^2 + 3t^3\right) \cdot (2t, 3t^2) = 6t^3 + 4t^4 + 6t^4 + 9t^5$ and work $= \int_1^4 6t^3 + 10t^4 + 9t^5 \, dt = \left[\frac{3}{2}t^4 + 2t^5 + \frac{3}{2}t^6\right]_{t=1}^{t=4} = 8571$

5: Find the work done by the vector field $\mathbf{F}(x, y, z) = (3x+2y+z, 2x+1y+3z, x+2y+3z)$ along the curves

(a) $g(t) = (t, t^2, t^3), 0 \leq t \leq 1$
Solution: $g'(t) = (1, 2t, 3t^2)$ and $\mathbf{F}(g(t)) = (3t + 2t^2 + t^3, 2t + t^2 + 3t^3, t + 2t^2 + 3t^3)$. Then
$F(g(t)) \cdot g'(t) = (3t + 2t^2 + t^3) + (4t^2 + 2t^3 + 6t^4) + (3t^3 + 6t^4 + 9t^5) = 3t + 6t^2 + 6t^3 + 12t^4 + 9t^5$.
Thus the work done is $\int_0^1 3t + 6t^2 + 6t^3 + 12t^4 + 9t^5 \, dt = \frac{89}{10}$.

(b) $\mathbf{g}(t) = (\cos t, \sin t, t), 0 \leq t \leq \pi/2$

Solution: $g'(t) = (-\sin t, \cos t, 1)$ and $\mathbf{F}(g(t)) = (3\cos t + 2\sin t + t, 2\cos t + \sin t + 3t, \cos t + 2\sin t + 3t)$ and
$F(g(t)) \cdot g'(t) = (-3\sin t\cos t - 2\sin t\sin t - t\sin t) + (2\cos t\cos t + \sin t\cos t + 3t\cos t) + (\cos t + 2\sin t + 3t)$.
The work done is the integral of this expression from 0 to. $\pi/2$ which is
$\left[2\sin t\cos t + \cos^2 t + \cos t + 3t\sin t + t\cos t\frac{3}{2}t^2\right]_0^{\pi/2} = -2 + \frac{3}{8}\pi^2 + \frac{3}{2}\pi$.

7: Find a potential function for $\mathbf{F}(x, y) = (2xe^y - \sin x \sin y, x^2 e^y + \cos x \cos y)$.

Solution: We want a function f such that $f_x(x, y) = 2xe^y - \sin x \sin y$ and $f_y(x, y) = x^2 e^y + \cos x \cos y$.

If we integrate the first component $2xe^y - \sin x \sin y$ with respect to x, we obtain a function $f(x, y) = x^2 e^y + \cos x \sin y + H(y)$ whose derivative with respect to y is the second component of $\mathbf{F}$ if $H'(y) = 0$ so choose $H(y) = 0$.

8: Discuss what happens when you try to find a potential function for $\mathbf{F} = (2xy + y^2, x^2 + 3xy)$
Solution: Start with integrating $2xy + y^2$ with respect to x. The result is $x^2 y + y^2 x + G(y)$ for some function G of y. Then the derivative of this expression with respect to y would be $x^2 + 2yx + G'(y)$ but there is no way to pick G so that this expression equals $x^2 + 3xy$.