MATH 223

Some Notes on Assignment 25

Exercises 28abcd, 30, 31, 32 and 33 of Chapter 6.

28abcd: Determine which of the following improper integrals converges or diverges:

(a) $\int_1^\infty \frac{1}{x}\,dx$

Solution: $\int_1^b \frac{1}{x}\,dx = \ln b - \ln 1 = \ln b$ and $\lim_{b\to\infty} \ln b = \infty$ so the integral diverges.

(b) $\int_1^\infty \frac{1}{x^2}\,dx$

Solution: $\int_1^b \frac{1}{x^2}\,dx = -\frac{1}{b} - (-\frac{1}{1}) = 1 - \frac{1}{b}$ and $\lim_{b\to\infty}(1-\frac{1}{b}) = 1$ so the integral converges.

(c) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Solution: $\int_1^b \frac{1}{\sqrt{x}}\,dx = 2\sqrt{b} - 2\sqrt{1} = 2\sqrt{b} - 1$ and $\lim_{b\to\infty} \ln b = \infty$ so the integral diverges.

(d) $\int_0^1 \frac{1}{x}\,dx$

Solution: $\int_a^1 \frac{1}{x}\,dx = \ln 1 - \ln a = -\ln a$ and $\lim_{a\to 0^+} \ln b = -\infty$ so the integral diverges.

30: Determine whether the given integrals are convergent or divergent. If convergent, find the value.

(a) $\int\int_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^{3/2}}\,dA.$

Solution: Switch to Polar Coordinates. The integral becomes

$$\int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} \frac{1}{(1+r^2)^{3/2}} r\,d\theta\,dr = \int_{r=0}^{r=\infty} 2\pi \frac{r}{(1+r^2)^{3/2}}\,dr = (2\pi)\lim_{b\to\infty}\int_{r=0}^{r=b}\frac{r}{(1+r^2)^{3/2}}\,dr$$

$$= (2\pi)\lim_{b\to\infty}\left[-(1+r^2)^{-1/2}\right]_{r=0}^{r=b} = (2\pi)\lim_{b\to\infty}\left[-\frac{1}{(1+b^2)^{1/2}} + \frac{1}{(1+0^2)^{1/2}}\right] = 2\pi\,[0+1] = 2\pi$$

(b) $\int\int\int_{\mathbb{R}^3} \frac{1}{(1+x^2+y^2+z^2)^{3/2}}\,dV$. (Optional Extra Credit)

31: Let R be the unit disk in $\mathbb{R}^2 = \{(x,y) : x^2+y^2 \le 1$. Determine whether the given integrals are convergent or divergent. If convergent, find the value.

(a) $\iint_R \frac{x^2}{((x^2+y^2)^{3/2}}\,dA$

Solution: Switch to polar coordinates. The integrand

$$\frac{x^2}{(x^2+y^2)^{3/2}} \text{ becomes} \frac{r^2\cos^2\theta}{(r^2)^{3/2}} = \frac{\cos^2\theta}{r}$$

and the integral becomes

$$\int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi}\frac{\cos^2\theta}{r} r\,d\theta\,dr = \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi}\cos^2\theta\,d\theta\,dr = \int_{r=0}^{r=1}\pi\,dr = \pi$$

(Recall $\int\cos^2\theta\,d\theta = \frac{\sin\theta\cos\theta+\theta}{2}$)

(b) $\iint_R \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\,dA$

Solution: Use polar coordinates again:

$$\iint_R \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\,dA = \iint_R \frac{\ln r^2}{r} r\,d\theta r = \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} 2\ln r\,d\theta\,dr = 4\pi\int_{r=0}^{r=1}\ln r\,dr$$

The integral is improper at $r=0$ so we examine $4\pi\int_{r=a}^{r=1}\ln r\,dr$ and then take the limit as $a\to 0^+$. Using integration by parts with $u=\ln r, dv = dr$, we find $\int \ln r\,dr = r\ln r - r$. Thus

$$\int_{r=a}^{r=1}\ln r\,dr = (1\ln 1 - 1) - (a\ln a - a) = -1 - a\ln a + a$$

and

$$\lim_{a\to 0^+}(-1+a) = -1 \text{ but } \lim_{a\to 0^+} a\ln a \text{ is an indeterminate } 0\times -\infty \text{ form.}$$

We use l'Hôpital's Rule with $a \ln a = \frac{\ln a}{1/a}$

$$\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{1/a} = \lim_{a\to 0^+} \frac{1/a}{-1/a^2} = \lim_{a\to 0^+} -a = 0$$

Thus $\int_{r=0}^{r=1} \ln r\, dr = -1$ and $\iint_R \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\, dA = -4\pi$

32: $\mathcal{I} = \iint_R \frac{1}{x^2+y^2}\, dx\, dy, R$ is interior of the unit square: $-1 < x < 1, -1 < x < 1$

Solution: Note that the unit disk D consisting of all points less than one unit from the origin lies entirely inside R so if the integral diverges on D, it will diverge on the larger set R as our function is always positive. The integral is improper because the function $f(x,y) = \frac{1}{x^2+y^2}$ is undefined at (0,0) and becomes unbounded as we approach the origin. We will evaluate the integral over D by switching to polar coordinates:
Note first: $\int_{r=a}^{r=1} \int_{\theta=0}^{\theta=2\pi} \frac{1}{r^2} r\, d\theta\, dr = 2\pi \int_{r=a}^{r=1} \frac{1}{r}\, dr = 2\pi(\ln 1 - \ln a) = -2\pi \ln a$

$$\text{Then } \mathcal{I} = \lim_{a\to 0^+} \int_{r=a}^{r=1} \int_{\theta=0}^{\theta=2\pi} \frac{1}{r^2} r\, d\theta\, dr = \lim_{a\to 0^+} -2\pi \ln a = \infty$$

so the integral diverges.

33: Let U be the set of points in $\mathbb{R}^3$ at least one unit from the origin; that is, $U = \{(x,y,z) : x^2+y^2+z^2 \geq 1\}$. Show that for $k > 5/2$, the triple integral

$$\mathcal{I} = \iiint_U \frac{(x^2+y^2)\ln(x^2+y^2+z^2)}{(x^2+y^2+z^2)^k}\, dV$$

converges and has value $\frac{16\pi}{3(2k-5)^2}$. Hint: Use spherical coordinates.

If $k > 5/2$, then $2k > 5$ so we can write $2k = 5 + p$ for some positive number p. Using spherical coordinates, the integral becomes

$$\iiint_U \frac{r^2 \sin^2\phi \ln r^2}{(r^2)^k} r^2 \sin\phi = \iiint_U \frac{r^4 \sin^3\phi \ln r^2}{r^{2k}} = \iiint_U \frac{r^4 \sin^3\phi (2\ln r)}{r^{5+p}} = \iiint_U 2\frac{\sin^3\phi \ln r}{r^{1+p}}$$

Now we'll put in the limits on the integral:

$$\mathcal{I} = 2\int_{r=1}^{r=\infty} \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=2\pi} \frac{\sin^3\phi \ln r}{r^{1+p}}\, d\theta\, d\phi\, dr = 2(2\pi) \int_{r=1}^{r=\infty} \int_{\phi=0}^{\phi=\pi/2} \frac{\sin^3\phi \ln r}{r^{1+p}} d\phi\, dr$$

For the integral with respect to ϕ, we have

$$\int \sin^3\phi\, d\phi = \int \sin\phi \sin^2\phi\, d\phi = \int \sin\phi(1-\cos^2\phi)\, d\phi = \int \left(\sin\phi - \sin\phi\cos^2\phi\right)\, d\phi = -\cos\phi + \frac{\cos^3\phi}{3}$$

so

$$\int_{\phi=0}^{\phi=\pi} \sin^3\phi\, d\phi = (-(-1) + -1/3) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

This result reduces our calculation of the integral to

$$\mathcal{I} = \frac{16\pi}{3} \int_{r=1}^{r=\infty} \frac{\ln r}{r^{1+p}} = \frac{16\pi}{3} \lim_{b\to\infty} \int_{r=1}^{r=b} \frac{\ln r}{r^{1+p}}$$

To integrate $\frac{\ln r}{r^{1+p}}$, use integration by parts with $u = \ln r, dv = r^{-1-p}$ so that $du = 1/r, v = r^{-p}/(-p)$.
Then $uv - \int v\, du = -(1/p)\frac{\ln r}{r^p} - (1/p^2)\frac{1}{r^p}$
and

$$\mathcal{I} = \frac{16\pi}{3} \lim_{b\to\infty} \left[\left(-(1/p)\frac{\ln b}{b^p} - (1/p^2)\frac{1}{b^p}\right) - \left((-1/p)\frac{\ln 1}{1^p} - (1/p^2)\frac{1}{1^p}\right)\right]$$

$$\mathcal{I} = \left[\frac{16\pi}{3}\left(\lim_{b\to\infty} -(1/p)\frac{\ln b}{b^p} - (1/p^2)\frac{1}{b^p}\right)\right] + \frac{16\pi}{3}\frac{1}{p^2}$$

but $\lim_{b\to\infty} \frac{1}{b^p}$ is easily seen to be 0 and $\lim_{b\to\infty} \frac{\ln b}{b^p} = 0$ by l'Hôpital's Rule since $p > 0$.

$$\text{Thus } \mathcal{I} = \frac{16\pi}{3}\frac{1}{p^2} = \frac{16\pi}{3p^2} = \frac{16\pi}{3(2k-5)^2} \text{ since } p = 2k-5$$