MATH 223

Some Notes on Assignment 24

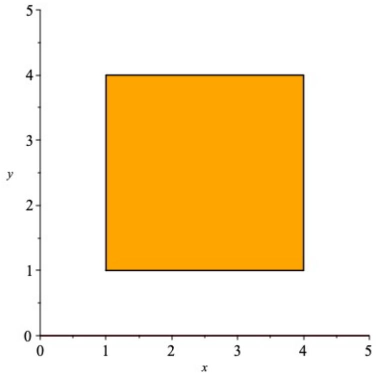
Exercises 21, 22, 24, 26 and 27 of Chapter 6.

21: Find the Jacobian of the transformation $T(u,v) = (u\cos v, u\sin v)$. For which points in the plane is T one-to-one?

Solution: Jacobian $= \begin{pmatrix} (u\cos v)_u & (u\cos v)_v \\ (u\sin v)_u & (u\sin v)_v \end{pmatrix} = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix}$
which has determinant $u\cos^2 v + u\sin^2 v = u$. Thus T is one-to-one at every point (u,v) where $u \neq 0$; that is, the vertical axis.

22: Define $T(u,v) = (u, v, u^v)$ for $u > 0$. Note that T maps a subset of $\mathbb{R}^2$ into a subset of $\mathbb{R}^3$.
(a) Determine the Jacobian matrix of this transformation.
(b) Let S be the square with vertices (1,1), (1, 4), (4,1), (4,4). What is the image of S under the transformation T?

Solution: (a) Jacobian $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ vu^{v-1} & u^v \ln u \end{pmatrix}$



The Square S

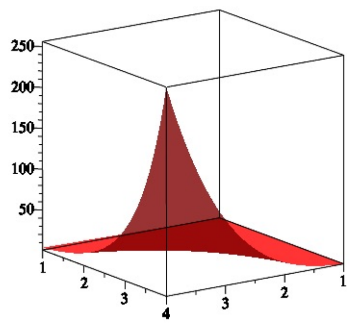
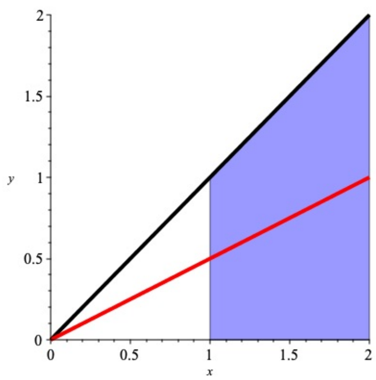


Image $(T(S))$

24: Let $\mathcal{R}$ be the region $\{(x,y) : 1 \le x \le 2, 0 \le y \le x\}$ and $f(x,y) = \frac{\sqrt{x^2+y^2}}{x}$. Find $\iint_{\mathcal{R}} f$ using the change of variable $x = r\cos\theta, y = r\sin\theta$.

Solution: Note the change of variables is the standard polar coordinate transformation. The figure below shows $\mathcal{R}$ as the shaded below region. The red line represents an arbitrary angle θ between 0 and $\pi/4$. The red line intersects the vertical line $x = 1$ at the point with polar coordinates $(r,\theta) = (\sec\theta, \theta)$ and it intersects the vertical line at $x = 2$ at the point with polar coordinates $(r,\theta) = (2\sec\theta, \theta)$. Thus in terms of polar coordinates the region $\mathcal{R}$ is described as $\{(r,\theta) : 0 \le \theta \le \pi/4, \sec\theta \le r \le 2\sec\theta\}$.



We also have

$$f(x,y) = \frac{\sqrt{x^2+y^2}}{x} = \frac{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}{r\cos\theta} = \sec\theta$$

Thus

$$\iint_{\mathcal{R}} f = \int_{\theta=0}^{\theta=\pi/4} \int_{r=\sec\theta}^{r=2\sec\theta} r\sec\theta\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi/4} \left(2\sec^3\theta - \frac{1}{2}\sec^3\theta\right) d\theta = \int_{\theta=0}^{\theta=\pi/4} \frac{3}{2}\sec^3\theta\, d\theta$$

which evaluates to $\frac{3}{4}\left(\ln(1+\sqrt{2})+\sqrt{2}\right)$.

26: Let A be the region in the first quadrant bounded by the curves $xy = 7, xy = 3, y^2 - x^2 = 9$ and $y^2 - x^2 = 16$ and $I = \iint_A (x^2+y^2)\, dA$. (a) Sketch the region A.
(b) Consider the transformation defined by $u = xy, v = y^2 - x^2$ and show the Jacobian of this transformation is $\begin{pmatrix} y & x \\ -2x & 2y \end{pmatrix}$.
(c) Show that the transformation sends A to the rectangular region in the (u,v)-plane with $3 \le u \le 7, 9 \le v \le 16$.
(d) Show that $I = \int_9^{16}\int_3^7 \frac{x^2+y^2}{2y^2+2x^2}\, du\, dv = 14$

Solution: (a) Figure 3 shows the region.
(b) Jacobian $= \begin{pmatrix} (xy)_x & (xy)_y \\ (y^2-x^2)_x & (y^2-x^2)_y \end{pmatrix} = \begin{pmatrix} y & x \\ -2x & 2y \end{pmatrix}$ which has determinant $2x^2+2y^2 = 2(x^2+y^2)$.
(c) The points in A satisfy $3 \le xy \le 7, 9 \le y^2 - x^2 \le 16$ which becomes $3 \le u \le 7, 9 \le v \le 16$. Figure 4 shows this rectangle in the uv-plane.
(d) Since the determinant of the Jacobian is $2(x^2+y^2)$, The change of variable formula shows $I = \iint_A (x^2+y^2)\, dA = \int_{u=3}^{u=7}\int_{v=9}^{v=16} \frac{1}{2}\, dv\, du = \frac{1}{2}$ Area of Rectangle $= \frac{1}{2}(4 \times 7) = 14$

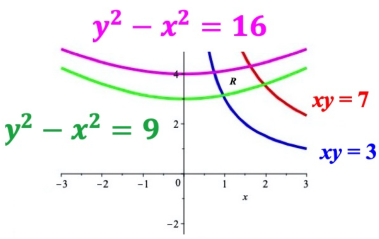


Figure 3

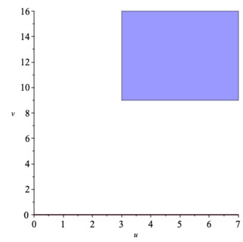


Figure 4

27: In the previous exercise, we did not solve for x and y in terms of u and v before computing the Jacobian matrix and its determinant. In this exercise, we outline how that approach might have worked.

1. Solve $u = xy, v = y^2 - x^2$ in the region A to show

$$x = \sqrt{\frac{-v+\sqrt{v^2+4u^2}}{2}}, y = \sqrt{\frac{v+\sqrt{v^2+4u^2}}{2}}$$

2. Find the Jacobian matrix for the transformation indicated in (a) and show that its determinant simplifies to

$$\frac{1}{2\sqrt{4u^2+v^2}}$$

Solution: Straightforward, but cumbersome differentiation shows the Jacobian matrix is

$$\begin{pmatrix} \frac{2u}{\sqrt{4u^2+v^2}\left(\sqrt{-2v+2\sqrt{4u^2+v^2}}\right)} & \frac{-2+\frac{2v}{\sqrt{4u^2+v^2}}}{4\left(\sqrt{-2v+2\sqrt{4u^2+v^2}}\right)} \\ \frac{2u}{\sqrt{4u^2+v^2}\left(\sqrt{2v+2\sqrt{4u^2+v^2}}\right)} & \frac{2+\frac{2v}{\sqrt{4u^2+v^2}}}{4\left(\sqrt{2v+2\sqrt{4u^2+v^2}}\right)} \end{pmatrix}$$

The determinant is

$$\frac{2u}{\sqrt{4u^2+v^2}\sqrt{-2v+2\sqrt{4u^2+v^2}}\sqrt{2v+2\sqrt{4u^2+v^2}}}$$

Note that the last two factors in the denominator are of the form $\sqrt{-A+B}\sqrt{A+B} = \sqrt{B^2-A^2}$ where $A = 2v$ and $B = 2\sqrt{4u^2+v^2}$ so $\sqrt{B^2-A^2} = \sqrt{4(4u^2+v^2)-4v^2} = \sqrt{16u^2} = 4u$. The determinant then simplifies to $\frac{2u}{4u\sqrt{4u^2+v^2}} = \frac{1}{2\sqrt{4u^2+v^2}}$

3. Verify that $x^2 + y^2 = \sqrt{4u^2 + v^2}$.
   Solution: $4u^2 + v^2 = 4(x^2y^2) + (y^2 - x^2)^2 = 4x^2y^2 + y^4 - 2x^2y^2 + x^4 = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$
4. Use this approach to show

$$\iint_A (x^2 + y^2)\, dA = \int_9^{16} \int_3^7 \frac{x^2 + y^2}{2y^2 + 2x^2}\, du\, dv$$

Solution: Apply Change of Variable Theorem