MATH 223

Some Notes on Assignment 18

Exercises 20, 21, 22, 23, and 24 in Chapter 5

20. Find and classify the critical points of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 8$

Solution A:(Without Lagrange Multipliers): Parametrize the constraint (circle of radius $\sqrt{8}$, center at the origin) as $x = \sqrt{8}\cos t, y = \sqrt{8}\sin t, 0 \le t \le 2\pi$. Then $f(x, y)$ becomes $\sqrt{8}\cos t\sqrt{8}\sin t = 8\sin t\cos t = 4\sin(2t)$ which has maximum value of 4 when $\sin 2t = 1$; that is $t = \pi/4$ or $t = 5\pi/4$. The minimum value is -4 which occurs at $t = 3\pi/4$ or $7\pi/4$. Maximum Values at $x = 2, y = 2$ or $x = -2, y = -2$. Minimum values when $x = 2, y = -2$ or $x = -2, y = 2$.

Solution B: Another Approach: Solve $x^2 + y^2 = 8$ for y in terms of x: $y = \sqrt{8 - x^2}$ and $y = -\sqrt{8 - x^2}$, then substitute each solution to get f as a function of x only and apply single variable calculus techniques for x on the interval from $-\sqrt{8}$ to $\sqrt{8}$.

Solution C: Use Lagrange Multiplier: Form

$$F(x, y, \lambda) = xy + \lambda(x^2 + y^2 - 8),$$

find gradient and set it equal to 0 to obtain 3 equations

$$y + 2x\lambda = 0, x + 2y\lambda = 0, x^2 + y^2 = 8.$$

Multiply the first equation by y and the second by x to show $2x^2\lambda = -xy = 2\lambda y^2$ so that $x^2 = y^2$. Thus $x = y$ or $x = -y$. When $x = y = 2$ or $x = y = -2$ we get the maximum; when $x = 2, y = -2$ or $x = -2, y = 2$, we get the minimum.

21: Use the method of Lagrange multipliers to solve Zoey's and Sydney's utility maximization problems of Examples 2 and 3 in Section 3.5.

Solution: Both problems involve the same constraint $x + \frac{y}{2} = D$ and both have utility functions of the form $kx^\alpha y^\beta$. Since k is positive for both Zoey and Sydney, each maximizes her utility by maximizing $x^\alpha y^\beta$.

Form the function

$$F(x, y, \lambda) = x^\alpha y^\beta + \lambda(x + \frac{y}{2} - D$$

Taking the gradient and setting the components equal to zero gives us three equations

$$(1)\ F_x = 0 : \alpha x^{\alpha-1}y^\beta + \lambda = 0$$

$$(2)\ F_y = 0 : \beta x^\alpha y^{\beta-1} + \frac{1}{2}\lambda = 0$$

$$(3)\ F_\lambda = 0 : x + \frac{y}{2} = D$$

If we multiply the second equation by 2, we find

$$\alpha x^{\alpha-1}y^\beta = -\lambda = 2\beta x^\alpha y^{\beta-1}.$$

Dividing through by $x^{\alpha-1}y^{\beta-1}$, we have

$$\alpha y = 2\beta x \text{ so } y = \frac{2\beta}{\alpha}x.$$

Use this equation for y in the equation $x + \frac{y}{2} = D$:

$$x + \frac{\beta}{\alpha}x = D \text{ so } x = \frac{\alpha}{\alpha+\beta}D,\ y = \frac{2\beta}{\alpha+\beta}D$$

For Zoey ($\alpha = \beta = 1/2$), the solution is $x = D/2,\ y = D$.
For Sydney $\alpha = 1/2, \beta = 1/5$), the solution is $x = 5D/7,\ y = 4D/7$. These choices will maximize utility. The other possible locations for extreme points are (0,2D) and (D,0), both of which give minimum utility of 0.

22: Abigail owns a pasta producing company. Her primary costs are labor ($15 per hour) and raw ingredients ($200 per ton). She estimates her revenue (in dollars) to be

$$f(x,y) = 180x^{\frac{2}{3}}y^{\frac{1}{3}}$$

if she uses x hours of labor and y tons of ingredients. If she has a budget constraint of $90,000, use the Lagrange multiplier approach to determine the optimal combination of labor and ingredients that will maximize her revenue.

Solution" Assessing the function Abigail has created to estimate her revenue in terms of labor and materials we see that the more she spends on either expense, the more revenue she will have. Therefore, Abigail will generate maximum revenue when she spends her entire budget. Given that her maximum budget is $\$90,000$, and that labor hours and tons of goods cost 15 dollars and 200 dollars respectively, Abigail's budget constraint is $15x + 200y - 90000 = 0$. To apply the method of Lagrange multipliers we combine the revenue function with the budget constraint to form a function of three variables with the form

$$F(x,y,\lambda) = 180x^{\frac{2}{3}}y^{1}3 - \lambda(15x + 200y - 90000).$$

We are interested in the critical points of this function so we must differentiate with respect to each variable and find the gradient of F.

$$\nabla F = \left(120\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} - 15\lambda, 60\frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} - 200\lambda, -15x - 200y + 90000\right)$$

At the critical point $\nabla F = (0,0,0)$ and the following equations will be true.

$$120\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} - 15\lambda = 0$$

$$60\frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} - 200\lambda = 0$$

$$-15x - 200y + 90000 = 0$$

Notice that the first two equations can be written in the form $80\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = 10\lambda$ and $3\frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}} = 10\lambda$ in which case we have

$$80\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = 3\frac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}}$$

Solving for x in terms of y we find $\frac{80}{3}y = x$. Substituting this expression for x into the third equation and solving for y we find $y = 150$ and $x = 4000$. Thus F has exactly one critical point at $(4000, 150)$. At this point, the level set $f(x,y) = k$ for maximum k is tangent to the budget constraint and Abigail's revenue is maximized. She should therefore pay for 4000 hours of labor and buy 150 tons of ingredients.

23: Anne is a psychological counselor whose practice is limited to two types of patients. One type requires a minutes of therapy per week and the other needs b hours weekly. She has a total of c hours available each week for therapy. If she has x clients of the first type and y of the second, she believes she will achieve a level of satisfaction measured by the function $S(x,y) = kx^{\frac{2}{3}}y^{\frac{1}{3}}$ for some positive constant k. Given the constraint on her time, use the method of Lagrange multipliers to find her optimal mixture of patients.
Solution: Just as in Abigail's problem, Anne has a function f she would like to maximize subject to a constraint function g. If we want to use the method of Lagrange Multipliers we write a new function F which has critical points precisely where $f(x,y)$ is maximized and $g(x,y) = 0$.

$$F(x,y,\lambda) = kx^{\frac{2}{3}}y^{\frac{1}{3}} - \lambda(ax + by - c)$$

The critical points of F occur wherever $\nabla F = (0,0,0)$ and the gradient of F is

$$\nabla F(x,y,\lambda) = \left(\frac{2ky^{\frac{1}{3}}}{3x^{\frac{1}{3}}} - \lambda a, \frac{kx^{\frac{2}{3}}}{3y^{\frac{2}{3}}} - \lambda b, c - ax - by\right).$$

When this gradient is the zero vector the following equations will be true:

$$1. \quad \frac{2ky^{\frac{1}{3}}}{3x^{\frac{1}{3}}} = \lambda a$$

$$2. \quad \frac{kx^{\frac{2}{3}}}{3y^{\frac{2}{3}}} = \lambda b$$

$$3. \quad ax + by = c$$

If we multiply equation 1 by b and equation 2 by a we get

$$\frac{2bky^{\frac{1}{3}}}{3x^{\frac{1}{3}}} = \frac{kax^{\frac{2}{3}}}{3y^{\frac{2}{3}}}.$$

Solving this equation for x in terms of y we find $x = \frac{2by}{a}$. Substituting this expression for x into equation iii we have $\frac{c}{3b} = y \Rightarrow \frac{2c}{3a} = x$. Having solved for both x and y in terms of the constants, we can substitute the results into equations 1 and 2 and solve for λ to find that the only critical point of F occurs at $\left(\frac{2c}{3a}, \frac{c}{3b}, \frac{k2^{\frac{2}{3}}}{3a^{\frac{2}{3}}b^{\frac{1}{3}}}\right)$. Anne's satisfaction function will be maximized and her time constraint will be met when $(x,y) = \left(\frac{2c}{3a}, \frac{c}{3b}\right)$.

24: Sasha operates a summer theater camp which enrolls M middle school students and H high school students. The cost of supplies (costumes, makeup, scripts, etc.) is \$$a$ for each middle schooler and \$$b$ per high school enrollee. If Sasha figures the number of parents who will attend the final performances is given by the function $P(x,y) = 22M^{\frac{3}{4}}S^{\frac{1}{4}}$ and he has a supply budget of \$$c$, what values of M and H will maximize P? What is the value of the Lagrange multiplier λ in the optimal solution?

Solution:. Applying the method of Lagrange Multipliers to the constraint function $P(x,y)$ we have $F(x,y,\lambda) = 22x^{\frac{3}{4}}y^{\frac{1}{4}} - \lambda(ax+by-c)$ which has gradient,

$$\nabla F = \left(22\frac{3}{4}\frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}} - \lambda a, 22\frac{1}{4}\frac{x^{\frac{3}{4}}}{y^{\frac{3}{4}}} - \lambda b, c - ax - by\right).$$

Wherever F has a critical point, P will be maximized and the constraint function will be satisfied. Wherever F has a critical point we have $\nabla F = (0,0,0)$ and

1. $22\frac{3}{4}\frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}} - \lambda a = 0$
2. $22\frac{1}{4}\frac{x^{\frac{3}{4}}}{y^{\frac{3}{4}}} - \lambda b = 0$
3. $c - ax - by = 0$

Multiply equation 1 by b and equation 2 by a and solving for x we find $\frac{3b}{a}y = x$. Substituting this value into equation 3 and solving for y we have $y = \frac{c}{4b}$. This value along with our expression for x in terms of y allows us to find $x = \frac{3c}{4a}$. Now that we have explicit expressions for the values of x and y we can substitute them into equations 1 and 2 find $\lambda = \frac{11}{2}3^{\frac{3}{4}}a^{-\frac{3}{4}}b^{\frac{1}{4}}$.