MATH 223

Some Notes on Assignment 14

Exercise 33 (old 26) in Chapter 4; Exercises 1 - 5 in Chapter 5.

33: Let Q be the positive first quadrant in the (u,v)-plane and define $\mathbf{f}(u,v) = \left(\sqrt{u^2+v^2}, \arctan\frac{v}{u}\right)$. Find the Jacobian matrix $J(u,v)$ and show that $\det J(u,v) = \frac{1}{\sqrt{u^2+v^2}}$ and The inverse of $J(u,v)$ is

$$\begin{pmatrix} \frac{u}{\sqrt{u^2+v^2}} & -v \\ \frac{v}{\sqrt{u^2+v^2}} & u \end{pmatrix}$$

Solution: Taking the partial derivatives with respect to u and v of both component functions of $f(u,v)$ we find $f_{1u} = \frac{u}{\sqrt{u^2+v^2}}$, $f_{1v} = \frac{v}{\sqrt{u^2+v^2}}$, $f_{2u} = \frac{-v}{u^2+v^2}$, and $f_{2v} = \frac{u}{u^2+v^2}$. The Jacobian matrix of f is

$$\mathbf{J}(u,v) = \begin{pmatrix} \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \\ \frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{pmatrix}.$$

From the formula of the determinant of a 2 by 2 matrix we have

$$\det \mathbf{J} = \frac{u^2}{\sqrt{u^2+v^2}(u^2+v^2)} + \frac{v^2}{\sqrt{u^2+v^2}(u^2+v^2)},$$

$$\det \mathbf{J} = \frac{1}{\sqrt{u^2+v^2}}.$$

On every point except the origin of the uv plane, the determinant of $\mathbf{J}$ is defined and the Jacobian matrix is invertible.
The given matrix is the inverse of $\mathbf{J}$ if and only if $\mathbf{J}\mathbf{J}^{-1} = \mathbf{I}$ where $\mathbf{I}$ is the 2 by 2 identity matrix. Multiplying $\mathbf{J}$ and $\mathbf{J}^{-1}$ we have

$$\mathbf{J}\mathbf{J}^{-1} = \begin{pmatrix} a_1 & b_2 \\ a_2 & b_2 \end{pmatrix}.$$

where the coordinates of the resultant matrix are:

$$a_1 = \frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} = 1$$

$$a_2 = \frac{-uv}{\sqrt{u^2+v^2}(u^2+v^2)} + \frac{uv}{\sqrt{u^2+v^2}(u^2+v^2)} = 0$$

$$b_1 = \frac{-vu}{\sqrt{u^2+v^2}} + \frac{uv}{\sqrt{u^2+v^2}} = 0$$

$$b_2 = \frac{v^2}{u^2+v^2} + \frac{u^2}{u^2+v^2} = 1$$

Substituting these results into the coordinates of the result matrix we have

$$\mathbf{J}\mathbf{J}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Chapter 5

1. a) Substituting x^2 for u and $x+5$ for v we have $g(f(x)) = x^3+5x^2$. The composition is a real-valued function of only a single variable; thus, we can differentiate using elementary calculus rules.

$$(g \circ f)'(x) = 3x^2 + 10x \Rightarrow (g \circ f)'(3) = 57.$$

b) To apply the Little Chain Rule, we must first find the gradient of g and the derivative of f. The gradient of g is $\nabla g = (v, u)$. The derivative of the vector-valued function f is $f'(x) = (2x, 1)$. The Little Chain Rule states that the derivative of the composition $(g \circ f)'(x)$ is $\nabla g(f(x)) \cdot f'(x)$. At $f(3) = (9, 8)$ we have

$$\nabla g(9, 8) \cdot f'(3) = (8, 9) \cdot (6, 1) = 57.$$

2. a) Substituting $\sin t$ for u, $\cos t$ for v, and $\tan t$ for w we get

$$(g \circ f)(t) = g(f(t)) = (\sin^3 t)(\cos t).$$

The composition is a real-valued function of a single variable and is therefore subject to single-dimensional differentiation rules.

$$(g \circ f)'(t) = 3(\sin^2 t)(\cos^2 t) - (\sin^4 t)$$

$$(g \circ f)'(\frac{\pi}{4}) = \frac{1}{2}$$

b) We must find the gradient of g, the derivative of f, and the value of $f(x)$ at $x = \frac{\pi}{4}$. The gradient of g is the 1 by 3 vector containing the partial derivatives with respect to u, v, and w: $\nabla g = (2uv^2w, u^2 2vw, u^2v^2)$. The derivative of f is the 3 by 1 vector containing the derivative of each component function with respect to t, that is, $f'(t) = (\cos t, -\sin t, \sec^2 t)$. At $t = \frac{\pi}{4}$ we have $f(\frac{\pi}{4}) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$. Applying the Little Chain Rule we get

$$(g \circ f)'\left(\frac{\pi}{4}\right) = \left((2)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right), \left(\frac{1}{2}\right)(2)\left(\frac{\sqrt{2}}{2}\right), \left(\frac{1}{4}\right)\right) \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\right) = \frac{1}{2}.$$

[3.] Note that f is function from $\mathbb{R}^2$ to $\mathbb{R}$ and g is a function from $\mathbb{R}$ to $\mathbb{R}^2$. The composition $(g \circ f)$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and its derivative is a 2 by 2 Jacobian Matrix. We should keep this in mind when assessing the validity of our answers.
Method 1: If we substitute x^3y^2 for t and write the composition as a single function we have $g(x, y) = (x^3y^2, e^{x^3y^2})$. The derivative of this function is a 2 by 2 matrix where each row contains the partial derivatives of the coordinate functions.

$$(g \circ f)' = g'(x, y) = \begin{pmatrix} 3x^2y^2 & 2x^3y \\ 3x^2y^2e^{x^3y^2} & 2x^3ye^{x^3y^2} \end{pmatrix} \Rightarrow g'(1, 2) = \begin{pmatrix} 12 & 4 \\ 12e^4 & 4e^4 \end{pmatrix}.$$

Method 2: The derivative of the composition can also be found using the General Chain Rule. The gradient of g is a 2 by 1 matrix of the derivatives of the component functions: $\nabla g = (1, e^t)$. The derivative of f is a 1 by 2 matrix containing partial derivatives with respect to x and y. If we evaluate $f(1, 2)$ and substitute the answer into the formula given by the General Chain Rule we have

$$\nabla g(f(1, 2)) \cdot f'(1, 2) = \begin{pmatrix} 1 \\ e^4 \end{pmatrix} \cdot \begin{pmatrix} 12 & 4 \end{pmatrix} = \begin{pmatrix} 12 & 4 \\ 12e^4 & 4e^4 \end{pmatrix}.$$

4 In this question, the composed function has its domain in $\mathbb{R}^2$ and its image in $\mathbb{R}^3$. We should then expect the derivative to be a 3 by 2 Jacobian Matrix.
Method 1: Substituting $f(x,y)$ for t in $g(t)$ we have

$$(g \circ f) = g(x,y) = (x+y, (x+y)^{\frac{3}{2}}, \sqrt{x+y}).$$

The derivative of this function is the 3 by 2 matrix of the partial derivatives of each component function.

$$(g \circ f)' = \begin{pmatrix} 1 & 1 \\ \frac{3}{2}\sqrt{x+y} & \frac{3}{2}\sqrt{x+y} \\ \frac{1}{2\sqrt{x+y}} & \frac{1}{2\sqrt{x+y}} \end{pmatrix}$$

$$(g \circ f)'(3,1) = \begin{pmatrix} 1 & 1 \\ 3 & 3 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Method 2: The gradient of g is $\nabla g = (2t, 3t^2, 1)$. The derivative of f is $f'(x,y) = \left(\frac{1}{2\sqrt{x+y}}, \frac{1}{2\sqrt{x+y}}\right)$. Evaluating f at $(x,y) = (3,1)$ we get $f(3,1) = 2$. The General Chain Rule says $(g \circ f)'(x,y) = g'(f(\mathbf{x})) \cdot f'(\mathbf{x})$. Substituting the above values into this equation gives

$$(g \circ f)'(3,1) = \begin{pmatrix} 4 \\ 12 \\ \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

5. The composition $(g \circ f)$ has both domain and image in $\mathbb{R}^2$, so its derivative is a 2 by 2 matrix.

Method 1: Rewriting the composition as a single function, $g : \mathbb{R}^2 \to \mathbb{R}^2$, gives $g(x,y) = (3x^2, 2x^4 - x^2y^2 - 4y^3)$. Taking the partial derivatives of each of these component functions we have

$$g'(x,y) = \begin{pmatrix} 6x & 0 \\ 8x^3 - 2xy^2 & -2x^2y - 4y^3 \end{pmatrix} \Rightarrow g'(3,4) = \begin{pmatrix} 18 & 0 \\ 120 & -328 \end{pmatrix}$$

Method 2: Differentiating g and f separately we have

$$g'(u,v) = \begin{pmatrix} 1 & 1 \\ v & u \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} 2x & -2y \\ 4x & 2y \end{pmatrix}$$

At $(x,y) = (3,4)$ we have $g(f(x,y)) = g(-7,34)$. Applying the General Chain Rule to find the derivative of the composition at $(3,4)$ we have

$$(g \circ f)'(3,4) = g'(-7,34) \cdot f'(3,4) = \begin{pmatrix} 1 & 1 \\ 34 & -7 \end{pmatrix} \cdot \begin{pmatrix} 6 & -8 \\ 12 & 8 \end{pmatrix} = \begin{pmatrix} 18 & 0 \\ 120 & -328 \end{pmatrix}.$$