MATH 223

Some Notes on Assignment 11

Exercises 17 and 18 in Chapter 4 and Problems A – C.

17. Show that the function f of one variable given by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } (x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable for all x but f' is not continuous at 0 so f is not continuously differentiable.

Solution: Wherever $x \neq 0$, $f(x)$ is the product and composition of differentiable functions, so it is differentiable. The interesting case is when $\frac{1}{x}$ is undefined at $x = 0$. To determine whether or not the function remains differentiable here we must inspect the limit of the difference quotient.

$$f'(0) = \lim_{x \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{x \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h}$$

if the limit exists. The function $\sin x$ is bounded above and below by 1, -1 for all x, so we can create the following upper and lower bounds:

$$-h \leq h \sin \frac{1}{h} \leq h \text{ so}$$

$$\lim_{h \to 0} -h \leq \lim_{h \to 0} h \sin \frac{1}{h} \leq \lim_{h \to 0} h$$

$$0 \leq \lim_{h \to 0} h \sin \frac{1}{h} \leq 0.$$

The derivative of f exists at at $x = 0$ and is equal to 0.

Using the rule for differentiation in the single-variable case we can find the derivative of f to be

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} : x \neq 0 \\ 0 : x = 0 \end{cases} .$$

For $f'(x)$ to be continuous, the limit of $f'(x)$ as x approaches 0 must be zero. That is,

$$\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

must equal 0; however, the first term in this limit will approach 0 as x gets small while the second term will achieve the values -1 and 1 on any neighbourhood of $x = 0$. The limit of the derivative of f does not then exist at $x = 0$.

18: Replace x^2 with x^3 in the previous exercise and determine if the resulting function is continuously differentiable everywhere.

Solution: To determine whether or not $f(x)$ is continuously differentiable we must first be certain that it is differentiable. In the case that $x \neq 0$, $f(x)$ is a product and composition of continuous functions and is therefore continuous. In the case that $x = 0$, we must make sure that the limit of the difference quotient exists.

$$f'(0) = \lim_{x\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{x\to 0} \frac{h^3 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h^2 \sin\frac{1}{h}$$

Just as in question 17, we can create a convergence of upper and lower bounds for this limit using the properties of the function $\sin x$.

$$\lim_{h\to 0} -h^2 \leq f'(0) \leq \lim_{h\to 0} h^2$$

$$0 \leq f'(0) \leq 0 \to f'(0) = 0$$

The function $f(x)$ is differentiable everywhere, but is its derivative continuous? Wherever $x \neq 0$, $f'(x)$ is a composition of continuous functions and is therefore continuous. When $x = 0$, the continuity becomes less clear. Inspecting the limit of $f'(x)$ as x approaches 0 we find

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}.$$

As we have shown in this question and the previous question, both terms in the right side of this equality are bounded and approach 0 as x gets very small, therefore, $\lim_{x\to 0} f'(x) = 0$ and $f(x)$ is continuously differentiable.

Problem A: For each of these functions f find gradient $\nabla f(\mathbf{x})$ of f at a general point in the domain of f:
(1) $f(x,y) = 2x^2?3y^2$: $\nabla f(x,y) = (f_x(x,y), f_y(x,y))\,(4x, -6y)$

(2) $f(x,y,z) = (5x-7y)z = 5xz-7yz$: $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)) = (5z, -7z, 5x-7y)$

(3) $f(x_1,x_2,x_3) = \frac{x_1x_3}{x_2}$: $\nabla f(x_1,x_2,x_3) = (f(_{x_1}, x_{x_2}, x_{x_3})) = \left(\frac{x_3}{x_2}, -\frac{x_1x_3}{x_2^2}, \frac{x_1}{x_2}\right)$

Problem B: Write an equation in terms of the coordinate variables (x, y, z) for the tangent hyperplane for $f(x,y,z) = 2x^2 - y^2 + 3z^2$ when $x = y = z = 1$.

Solution: $f_x(x,y,z) = 4x$, $f_y(x,y,z) = -2y$, $f_z(x,y,z) = 6z$ so $f_x(1,1,1) = 4$, $f_y(1,1,1) = -2$, $f_z(1,1,1). = 6z$. Since $f(1,1,1) = 2-1+3 = 4$, an equation for the hyperplane is

$$w = 4 + 4(x-1) - 2(y-1) + 6(z-1)$$

Problem C: Let f be the real-valued function $f : \mathbb{R}^p \to \mathbb{R}^1$ defined by $f(\mathbf{x}) = |\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^p$. If $p = 2$, prove that $\nabla f(\mathbf{x}) = 2\mathbf{x}$. Is this result true for other values of p?

Solution: For $p = 2$, $f(\mathbf{x}) = f(x,y) = (x,y)\cdot(x,y) = x^2 + y^2$ so that $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$. Thus the gradient of f is $(2x, 2y) = 2(x,y) = 2\mathbf{x}$

For $p > 2$, $f(\mathbf{x}) = \mathbf{x}\cdot\mathbf{x} = (x_1, x_2, ..., x_p)\cdot(x_1, x_2, ..., x_p) = x_1^2 + x_2^2 + ... + x_p^2$. Hence $f_{x_i}(\mathbf{x}) = 2x_i$ for each i.Thus the gradient of f is $(2x_1, 2x_2, ..., 2x_i, ..., 2x_p) = 2\mathbf{x}$.