MATH 223

Some Notes on Assignment 10

Exercises 3, 7, 10, 15, and 16 in Chapter 4.

3. In our proof of part (e) of Theorem 4.1.1, we claimed that $|g(\mathbf{x}) - M| < \frac{|M|}{2}$ implies that $|g(\mathbf{x})| > \frac{|M|}{2}$. Show that this claim is true.
Solution: Suppose that $|M - g(\mathbf{x})| < \frac{|M|}{2}$. Note that, $|M - g(\mathbf{x})| = |g(\mathbf{x}) - M|$. The inequality is then equivalent to

$$|M - g(\mathbf{x})| < \frac{|M|}{2}.$$

Adding the absolute value of $g(\mathbf{x})$ to both sides we find

$$|M - g(\mathbf{x})| + |g(\mathbf{x})| < \frac{|M|}{2} + |g(\mathbf{x})|.$$

Using the Triangle Inequality on the left hand side of the inequality we get

$$|M| \leq |M - g(\mathbf{x})| + |g(\mathbf{x})| < \frac{|M|}{2} + |g(\mathbf{x})|$$

$$\frac{|M|}{2} < |g(\mathbf{x})|.$$

7. A naturally occurring idea is that a vector limit should be the same as an iterated limit; e.g., $\lim_{(x,y)\to(a,b)} f(x,y)$ should be the same as what we would get by first letting x approach a and then letting y approach b. Consider $f(x,y) = \frac{xy}{x^2+y^2}$, which shows the vector limit does not always behave this way.

1. Show $\lim_{x\to 0}(\lim_{y\to 0} f(x,y)) = 0$.
2. Show $\lim_{y\to 0}(\lim_{x\to 0} f(x,y)) = 0$.
3. Show $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution: In the case that we want to take an iterative limit of a vector-valued function, we can treat all the variables not under inspection as constants and focus solely on a single variable.
a) If we first take the limit with respect to y and then take the limit with respect to x we find

$$\lim_{x\to 0}(\lim_{y\to 0}(\frac{xy}{x^2+y^2}))$$

$$\lim_{x\to 0}(\frac{0}{x^2}) = 0.$$

b) If we first take the limit with respect to x and the limit with respect to y we get

$$\lim_{y\to 0}(\lim_{x\to 0}(\frac{xy}{x^2+y^2}))$$

$$\lim_{y\to 0}(\frac{0}{y^2}) = 0.$$

c) Now if we attempt to find the limit of f as x and y approach $(0, 0)$ simultaneously, we are able to find different answers depending upon the route we take towards the origin. Consider the limit as (x, y) approaches the origin along the line $x = y$.

$$\lim_{(x,y)\to(0,0)} (\frac{xy}{x^2 + y^2}) = \lim_{(x,y)\to(0,0)} (\frac{x^2}{2x^2}) = \frac{1}{2}$$

This is different from the limit we get if we approach the origin along the line $x = -y$.

$$\lim_{(x,y)\to(0,0)} (\frac{xy}{x^2 + y^2}) = \lim_{(x,y)\to(0,0)} (\frac{-x^2}{2x^2}) = -\frac{1}{2}$$

The limit only exists in the case that it is independent of the route taken towards the point of inspection; thus, the limit of f as (x, y) heads to $(0, 0)$ does not exists.

10 For the real-valued function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

show

1. the limit of f as $\mathbf{h}$ goes to $\mathbf{0}$ along the x or y axis is 0,
2. the limit of f as $\mathbf{h}$ goes to $\mathbf{0}$ along any straight line through the origin is also 0. [Let $\mathbf{h} = (x, mx)$],
3. but the limit of f as $\mathbf{h}$ goes to $\mathbf{0}$ along the parabola $y = x^2$ is $\frac{1}{2}$.
4. Explain why
$$\lim_{(x,y)\to(0,0)} f(x,y)$$
does not exist.

Solution: a) To find the limit of the function as (x, y) approaches the origin along the x or y axis we can fix one of the variables to be 0 and take the limit as the other variable approaches 0. Inspecting along the y axis we have

$$\lim_{y\to 0} \frac{0^2 y}{0^4 + y^2} = 0.$$

If y is fixed to be 0 and the limit is taken as (x, y) head to the origin along the x we get

$$\lim_{x\to o} \frac{x^2 0}{x^4 + y^2} = 0.$$

b) If we substitute mx for y we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^2(x^2 + m)} = \lim_{(x,y)\to(0,0)} \frac{mx}{x^2 + m} = 0.$$

c) Now if we wish to examine the limit as (x, y) approaches the origin along the parabola $y = x^2$ we need only substitute x^2 for y to find

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2}.$$

d) The limit of $f(x,y)$ as (x,y) approaches the origin is dependent on the path travelled; thus, the limit does not exist.

15. Suppose f and g are real-valued functions of n variables which are differentiable at all points of $\mathcal{R}^n$. Show that

1. $f+g$ and
2. af for any constant a

are differentiable on all of $\mathcal{R}^n$.
Solution: 15. a) If f and g are both differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ then there exist 1 by n matrices $\mathbf{m}_f$, $\mathbf{m}_g$ such that

$$\lim_{|\mathbf{h}|\to 0} \frac{f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\mathbf{m}_f\mathbf{h}}{|\mathbf{h}|} + \lim_{|\mathbf{h}|\to 0} \frac{g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)-\mathbf{m}_g\mathbf{h}}{|\mathbf{h}|} = 0.$$

Theorem 4.1.1. part a says that the sum of these two limits is the limit of their sums.

$$\lim_{|\mathbf{h}|\to 0} \frac{(f(\mathbf{x}_0+\mathbf{h})+g(\mathbf{x}_0+\mathbf{h}))-(f(\mathbf{x}_0)+g(\mathbf{x}_0))-(\mathbf{m}_f\mathbf{h}+\mathbf{m}_g\mathbf{h})}{|\mathbf{h}|} = 0$$

$$\lim_{|\mathbf{h}|\to 0} \frac{(f+g)(\mathbf{x}_0+\mathbf{h})-(f+g)(\mathbf{x}_0)-(\mathbf{m}_f+\mathbf{m}_g)\mathbf{h}}{|\mathbf{h}|} = 0$$

If $\mathbf{m}$ is the the 1 by n matrix equal to the sum $\mathbf{m}_f+\mathbf{m}_g$, then this equality is precisely the limit of the difference quotient which ensures differentiability of $f+g$ at an arbitrary point $\mathbf{x}_0$.

b) If f is a a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$ then there exists a 1 by n matrix $\mathbf{m}_f$ such that

$$\lim_{|\mathbf{h}|\to 0} \frac{f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\mathbf{m}_f\mathbf{h}}{|\mathbf{h}|} = 0.$$

If both sides of this equality are multiplied by α they become

$$\alpha \cdot \lim_{|\mathbf{h}|\to 0} \frac{f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\mathbf{m}_f\mathbf{h}}{|\mathbf{h}|} = 0.$$

As stated in Theorem 4.1.1. part c, the scalar multiple of the limit of a continuous function is the limit of that function's multiple.

$$\lim_{|\mathbf{h}|\to 0} \frac{\alpha f(\mathbf{x}_0+\mathbf{h})-\alpha f(\mathbf{x}_0)-\alpha\mathbf{m}_f\mathbf{h}}{|\mathbf{h}|} = 0$$

If $\alpha\mathbf{m}_f=\mathbf{m}$ then this equality ensures that the limit of the difference quotient for $\alpha f(\mathbf{x}_0)$ exists, and that αf is differentiable at an arbitrary point $\mathbf{x}_0$.

16. Show that the set $\mathcal{L}$ of all real-valued functions differentiable on $\mathcal{R}^n$ is a vector space.
Solution:To show that the set $\mathcal{L}$ forms a vector space we must show that it is closed under scalar multiplication and element addition. That is, if f and g are real-valued functions on $\mathbb{R}^n$ and α is a scalar, then $f+g$ and αf are included in $\mathcal{L}$.
i) For the function $f+g$ to be differentiable, there must by a matrix $\mathbf{m}$ satisfying the

limit of the difference quotient at any arbitrary point $\mathbf{x}$. That is, there must be an $\mathbf{m}$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{(f+g)(\mathbf{x}+\mathbf{h})-(f+g)(\mathbf{x})-\mathbf{mh}}{|\mathbf{h}|}=0.$$

Because $(f+g)(\mathbf{x})=f(\mathbf{x})+g(\mathbf{x})$ we can expand the numerator on the left side to get

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{(f)(\mathbf{x}+\mathbf{h})-f(\mathbf{x})+g(\mathbf{x}+\mathbf{h})-g(\mathbf{x})-\mathbf{mh}}{|\mathbf{h}|}.$$

Now letting $\mathbf{m}$ be $\nabla f+\nabla g$ this limit becomes

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{(f)(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-\nabla f\mathbf{h}}{|\mathbf{h}|}+\lim_{\mathbf{h}\to\mathbf{0}} \frac{(g)(\mathbf{x}+\mathbf{h})-g(\mathbf{x})-\nabla g\mathbf{h}}{|\mathbf{h}|}.$$

By the differentiability of f and g, the sum of these limits exists and is equal to 0. Thus, $f+g$ is differentiable and included in the set $\mathcal{L}$.
ii) For the function αf to be included in the set $\mathcal{L}$, it too must be differentiable. Because f is differentiable, we know there exists a matrix $\mathbf{m}$ such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{(f)(\mathbf{x}+\mathbf{h})-f(\mathbf{x})-\mathbf{mh}}{|\mathbf{h}|}=0.$$

If we multiply both sides of this equality we have

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\alpha f(\mathbf{x}+\mathbf{h})-\alpha f(\mathbf{x})-\alpha\mathbf{mh}}{|\mathbf{h}|}=0.$$

Notice that this limit being equal to 0 is a sufficient condition for proving that the function αf is differentiable at an arbitrary point $\mathbf{x}$ with gradient $\nabla(\alpha f)=\alpha\nabla f$.