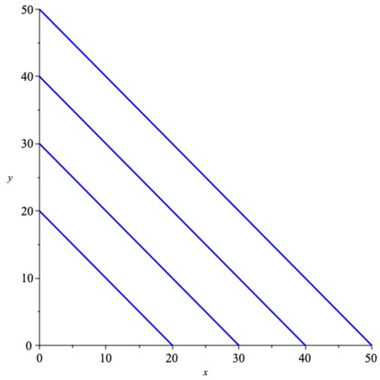
MATH 223

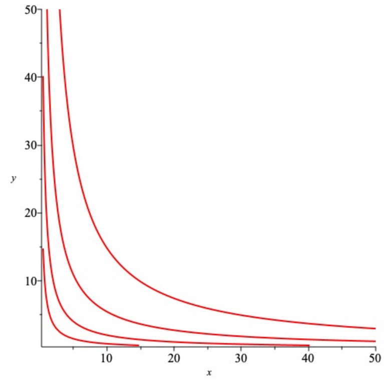
Some Notes on Assignment 9

Exercises 49ab, 51, 52a, 31 and 33 in Chapter 3.

49ab. Graph a set of indifference curves for the utility functions specified:
Solution:



(a) Curves of indifference for $u(x, y) = x + y$



(b) Curves of indifference for $u(x, y) = \ln xy$.

51. With a total of \$D to spend on apples and bananas, what combination will maximize Sydney's utility? *Solution*: From example 2 in section 3.5.1 we have Sydney's utility function for x 1 dollar apples and y 50 cent bananas is $s(x, y) = 15\sqrt{x}\sqrt[5]{y}$. If Sydney is spending D total dollars on apples and bananas then we have $D = x + \frac{y}{2}$. If we solve for x, we can then write the utility function in terms of y only.

$$D = x + \frac{y}{2} \to D - \frac{y}{2} = x, s(x, y) = 15\sqrt{x}\sqrt[5]{y} \to s(y) = 15\sqrt{D - \frac{y}{2}}\sqrt[5]{y}$$

The constrained utility function $s(y)$ will be maximized when its derivative is 0 and its second derivative is negative; however, $s(y)$ is also maximized when the square of $\sqrt{D - \frac{y}{2}}\sqrt[5]{y}$ is maximized. Instead of finding a more complicated derivative, we can differentiate $f = (D - \frac{y}{2})(y^{\frac{2}{5}}) = Dy^{\frac{2}{5}} - (\frac{1}{2})y^{\frac{7}{5}}$ to find the optimal combination of apples and bananas.

$$f' = D(\frac{2}{5})y^{-\frac{3}{5}} - \frac{7}{10}y^{\frac{2}{5}}$$

$$f'' = D(-\frac{6}{25})y^{-\frac{8}{5}} - (\frac{14}{50})y^{-\frac{3}{5}}$$

The second derivative f'' will be negative for all positive values of D and y, so function is concave down everywhere and any point at which $f' = 0$ will be a maximum.
Letting the first derivative be equal zero we get

$$D(\frac{2}{5})y^{-\frac{3}{5}} = (\frac{7}{10})y^{\frac{2}{5}}$$

$$D(\frac{2}{5}) = (\frac{7}{10})y \to (\frac{4}{7})D = y$$

Now that we have an optimal value of y in terms of D we can solve for the optimal value of x.

$$D = x + \frac{y}{2} \to D = x + (\frac{2}{7})D$$

$$x = (\frac{5}{7})D$$

52a. Find the marginal rate of substitution for Zoey's and Sydney's utility functions.
Solution:a) Zoey's utility function is $z(x, y) = \sqrt{xy}$. Then the marginal utility of x is $z_x = (\frac{y}{2})(xy)^{-\frac{1}{2}}$. The marginal utility of y is $z_y = (\frac{x}{2})(xy)^{-\frac{1}{2}}$.
Sydney's utility function is $s(x, y) = 15\sqrt{x}(y^{\frac{1}{5}})$. The marginal utility of x is $s_x = (\frac{15}{2})y^{\frac{1}{5}}x^{-\frac{1}{2}}$. The marginal utility of y is $s_y = 3x^{\frac{1}{2}}y^{-\frac{4}{5}}$.

31. Extend the result of Clairaut's Theorem to show that under appropriate continuity assumptions, we have $f_{xyx} = f_{xxy} = f_{yxx}$.
Solution: Suppose we have a continuous function f for which all first, second, and third order partial derivatives are continuous. By Clairaut's Theorem, $f_{xy} = f_{yx}$. If we differentiate both sides of this equality with respect to x we get

$$\frac{d}{dx}f_{xy} = \frac{d}{dx}f_{yx}$$

$$f_{xyx} = f_{yxx} \text{ (1)}$$

Now let $g = f_x$. Because all second and third order partial derivatives of f are continuous, all first and second order partial derivatives of g are continuous. This continuity is sufficient for us to apply Clairaut's Theorem to g and find $g_{xy} = g_{yx}$. If we substitute f_x in for g we have

$$g_{xy} = g_{yx} \rightarrow f_{xxy} = f_{xyx} \text{ (2)}.$$

Combinging results (1) and (2) we have

$$f_{xxy} = f_{xyx} = f_{yxx}.$$

33. Consider the function of two variables defined by $f(x, y) = 2xy\frac{x^2-y^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ with $f(0, 0) = 0$. Using the definition of partial derivatives, determine $f_x(0, 0)$ and $f_y(0, 0)$. Show that $f_{xy}(0, 0) = -2$ but $f_{yx}(0, 0) = +2$ so the mixed partials are not equal at the origin. Explain why Clairaut's Theorem does not apply to this function. .
Solution: The definition of the partial derivative with respect to x of $f(x, y)$ is

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Examining this limit at the origin we find

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{f(h, 0)}{h} = 0,$$

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{f(0, h)}{h} = 0.$$

To evaluate the mixed partial derivatives at the origin we first need to find general expressions for f_x and f_y at any arbitrary point (x, y). Applying the Product Rule to the numerator and the Quotient Rule to the entire expression we can solve for the general partial derivatives.

$$f_x = \frac{2y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y = \frac{2x(x^4 - 4x^2y^2 - {y^4}^2}{(x^2+y^2)^2}$$

Now we can apply the definition of the partial derivative at the origin to find the values of the mixed partials at the origin.

$$f_{xy}(0,0) = \lim_{h\to 0}(\frac{1}{h})(f_x(0, 0+h) - f_x(0,0)) = \lim_{h\to 0}(\frac{1}{h})(f_x(0,h))$$

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{-2h^5}{h^5} = -2$$

Differentiating f_y with respect to x we have

$$f_{yx}(0,0) = \lim_{h\to 0}(\frac{1}{h})(f_y(0+h, 0) - f_x(0,0)) = \lim_{h\to 0}(\frac{1}{h})(f_x(h,0))$$

$$f_{yx}(0,0) = \lim_{h\to 0}\frac{2h^5}{h^5} = 2.$$

Thus the mixed partial derivatives are not equivalent at the origin. Applying Clairaut's Theorem to a function f requires f, the first, and the second order partial derivatives to all be continuous over the interval of inspection. Neither of the first order partial derivatives of f are continuous at the origin, so Clairaut's Theorem can't guarantee anything about the mixed partial derivatives.