MATH 223

Some Notes on Assignment 7

Chapter 3: 24 abc, 25abc, 26ac, 27 and 29.

24abc: Find f_x and f_y or each of the following:
a: $f(x,y) = \sin xy$
Solution: To take the partial derivative with respect to x we can treat y as a constant and the derivative becomes a simple application of the Chain Rule: $f_x(x,y) = y\cos xy, f_y(x,y) = x\cos xy$
b: $f(x,y) = \tan e^x$
Solution: Recall from single variable calculus that the derivative of $\tan x$ is $\sec^2 x$ and the derivative of e^x is e^x. If we hold y constant and take the derivative with respect to x, the Chain Rule gives $f_x(x,y) = \frac{e^x}{(\cos^2 e^x)}$ As there is no y included in $f(x,y)$, the entire expression is a constant if x is treated as a constant. The derivative $f_y(x,y)$ is then 0.

c: $f(x,y) = \frac{\arctan y}{x}$ *Solution:* If we treat y as a constant then the term $\arctan y$ is a constant. Then the partial derivative with respect to x is $(\arctan y)(-x^{-2})$. From single variable calculus we know that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$. The partial derivative of $f(x,y)$ with respect to y is then $(\frac{1}{x})\frac{1}{1+y^2}$.

25. a) From question 24a we have $f_x(x,y) = y\cos xy$ and $f_y(x,y) = x\cos xy$. Both of these expressions can be differentiated using applications of the Chain Rule and Product Rule.

$$f_{xx}(x,y) = -y^2 \sin xy, f_{yy}(x,y) = -x^2\sin(yx)$$

$$f_{xy}(x,y) = f_{yx} = \cos yx - yx\sin yx$$

The second order partial derivatives f_{xy} and f_{yx} are equal.
b) From question 24b we have $f_x(x,y) = \frac{e^x}{\cos^2 e^x}$ and $f_y(x,y) = 0$. As there is no y term in either expression, the second order partial derivatives f_{xy}, f_{yx} are both 0. To find f_{xx} we can differentiate f_x by applying the Quotient or Product Rules along with the Chain Rule to find

$$f_{xx}(x,y) = \frac{e^x(2e^x \sin e^x + \cos e^x)}{\cos^3 e^x}$$

c) From question 24c we have $f_x(x,y) = (\arctan y)(-x^{-2})$ and $f_y(x,y) = (\frac{1}{x})(\frac{1}{1+y^2})$. Differentiating f_x with respect to x gives $f_{xx}(x,y) = 2\arctan y(x^{-3})$. To find f_{xy}, we need only differentiate the term $\arctan y$ and multiply by the

x term to find $f_{xy}(x,y) = (\frac{1}{y^2+1})(-x^{-2})$. We can differentiate the x term of f_y and multiply by the constant y term to get $f_{yx}(x,y) = (\frac{1}{y^2+1})(-x^{-2})$. Notice that this is the same as f_{xy}. Differentiating the y term of f_y and multiplying by the constant x term gives $f_{yy}(x,y) = (\frac{1}{x})\frac{-2y}{(y^2+1)^2}$.

26ac For each of these functions f, determine $f_x(2,3)$ and $f_y(2,3)$:

a: $f(x,y) = x^3 + 4xy - y^2$
Solution: If $f(x,y) = x^3 + 4xy - y^2$ then we have

$$f_x(x,y) = 3x^2 + 4y \to f_x(2,3) = 24 \text{ and}$$

$$f_y(x,y) = 4x - 2y \to f_y(2,3) = 2$$

c $f(x,y) = \frac{2x-3y}{3x+2y}$ *Solution:* Because $f(x,y) = \frac{2x-3y}{3x+2y}$ contains an x term and y term in both the numerator and the denominator, we will need to apply The Quotient Rule to find both partial derivatives.

$$f_x(x,y) = \frac{13y}{(3x+2y)^2} \to f_x(2,3) = \frac{13}{48} \text{ and}$$

$$f_y(x,y) = \frac{-13x}{(3x+2y)^2} \to f_y(2,3) = \frac{-13}{72}$$

27: Josh's utility function for two particular goods has the form $U(x,y) = (x+3)^2(y+2)^3$. Find the marginal utility functions U_x and U_y and evaluate them if $x = 4, y = 4$.
Solution: If the utility of goods x and y is $U(x,y) = (x+3)^2(y+2)^3$ then the marginal utility of good x is $U_x(x,y) = (y+2)^3 2(x+3)$ and the marginal utility of good x at $(4,4)$ is 3024. The marginal utility of good y is $u_y(x,y) = (x+3)^2 3(y+2)^2$; evaluated at $(4,4)$ we have $U_y(4,4) = 5292$.

29: A thin, homogeneous metal rod lying along the horizontal axis from 0 to L has a nonuniform temperature. Heat (thermal energy) transfers from regions of higher temperature to regions of lower temperature. Under certain conditions the function $u(x,t)$ which gives the temperature at position x and time t obeys the **diffusion equation** $u_{xx} = 4u_t$. Show that function

$$u(x,t) = \frac{e^{-x^2/t}}{\sqrt{t}}, t > 0$$

satisfies the diffusion equation.

Solution: To see whether or not the function $u(x,t)$ satisfies the diffusion equation we need to find the second order partial derivative u_{xx} and the first order partial derivative u_t. If we factor out $\frac{1}{\sqrt{t}}$ and differentiate with respect to x using the Product Rule we find

$$u_x(x,t) = \frac{-2xe^{\frac{-x^2}{t}}}{t^{\frac{3}{2}}}.$$

Now we can factor out $\frac{1}{t^{\frac{3}{2}}}$ and use the Product Rule to find

$$u_{xx}(x,t) = e^{\frac{-x^2}{t}}(\frac{4x^2}{t^{\frac{5}{2}}} - \frac{2}{t^{\frac{3}{2}}}).$$

Now we need to use the Quotient Rule on $u(x,t)$ to find $u_t(x,t)$.

$$u_t(x,t) = e^{\frac{-x^2}{t}}(\frac{x^2}{t^{\frac{5}{2}}} - \frac{1}{2t^{\frac{3}{2}}}) = 4u_{xx}(x,t).$$

The equation does satisfy the diffusion equation.