

MATH 223

Some Notes on Assignment 4

Chapter 2: Exercises 35, 40, 42, 43 and Problem A

35: For a parametrized curve $\mathbf{x}(t)$, $a \leq t \leq b$, the **arc length** \mathbf{L} is the integral of its speed:

$$\mathbf{L} = \int_a^b |\mathbf{v}(t)| dt = \int_a^b |\mathbf{x}'(t)| dt$$

(a): If the speed is constant, show that arc length is given by the classic distance = rate \times time formula.

Solution: If speed is constant, then $|\mathbf{x}'(t)| = r$ for some constant r in which case $\mathbf{L} = \int_a^b r dt = r(b - a) = \text{rate} \times \text{time}$.

(b): Show that the curve with parametrization $\mathbf{g}(t) = (a \sin \omega t, a \cos \omega t)$ with $a > 0$ traces out, in a clockwise direction a portion of the circle of radius a centered at the origin with speed $a\omega$. We call the constant ω the **angular speed**.

Solution: $|\mathbf{g}(t)| = \sqrt{a^2 \sin^2 \omega t + a^2 \cos^2 \omega t} = \sqrt{a^2} = a$ so $\mathbf{g}(t)$ lies on the circle of radius a centered at the origin for all t . At $t = 0$, we are at the point $(0, a)$; as t increases from 0, $a \sin \omega t$ will increase and $a \cos \omega t$ will decrease so we move in a clockwise direction. We trace out the entire circle as t goes from 0 to 2π .

Now speed = $|\mathbf{g}'(t)| = |(a\omega \cos \omega t, -a\omega \sin \omega t)| = \sqrt{a^2 \omega^2 \cos^2 \omega t + a^2 \omega^2 \sin^2 \omega t} = \sqrt{a^2 \omega^2} = a\omega$.

(c) Describe the difference in motion between the parametrizations $\mathbf{x}(t) = (\sin t, \cos t)$, $0 \leq t \leq 2\pi$ and $\mathbf{y}(u) = (\sin u, \cos u)$, $0 \leq u \leq 6\pi$.

Solution: Both trace out the unit circle in the same direction and with the same speed but \mathbf{y} traces out the circle 3 times while \mathbf{x} only traces it out once.

40: For the position function $\mathbf{p}(t) = (\cos t, \sin t)$ with velocity \mathbf{v} and acceleration \mathbf{a} , what is the geometric relationship between \mathbf{p} and \mathbf{v} ? Between \mathbf{p} and \mathbf{a} ? Between \mathbf{v} and \mathbf{a} ? *Solution:* $\mathbf{v} = (-\sin t, \cos t)$ and $\mathbf{a} = (-\cos t, -\sin t) = -\mathbf{p}$ so \mathbf{p} and \mathbf{v} are orthogonal to each other and \mathbf{a} points in the opposite direction of \mathbf{p} so \mathbf{a} and \mathbf{v} are also orthogonal.

42: Find all differentiable vector valued functions \mathbf{p} such that $\mathbf{p}'(t) = (te^t, \cos t, 2t\sqrt{1+t^2})$ and $\mathbf{p}(0) = (0, 0, 0)$

Solution : If $\mathbf{p}'(t) = (te^t, \cos t, 2t\sqrt{1+t^2})$ then $\mathbf{p}(t)$ have the form $\mathbf{p}(t) = ((t-1)e^t + C_1, \sin t + C_2, \frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3)$ where $C_1, C_2,$ and C_3 are all real constants. Given that we know $\mathbf{p}(0) = (0, 0, 0)$ we can solve for the undetermined constants.

$$(0-1)e^t + C_1 = 0 \Rightarrow C_1 = 1$$

$$\sin 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$\frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3 = 0 \Rightarrow C_3 = -\frac{2}{3}$$

Hence $\mathbf{P}(t) = ((t-1)e^t + 1, \sin t, \frac{2}{3}(1+t^2)^{\frac{3}{2}} - \frac{2}{3})$.

42: If $\mathbf{p}'(t) = (te^t, \cos t, 2t\sqrt{1+t^2})$ then $\mathbf{p}(t)$ have the form $\mathbf{p}(t) = ((t-1)e^t + C_1, \sin t + C_2, \frac{2}{3}(1+t^2)^{\frac{3}{2}} + C_3)$ where $C_1, C_2,$ and C_3 are all real constants. [Use integration by parts on the first component with $u = te^t, dv = dt,$ and substitution $u = 1+t^2$ for the third component.

Given that we know $\mathbf{p}(0) = (0, 0, 0)$ we can solve for the undetermined constants.

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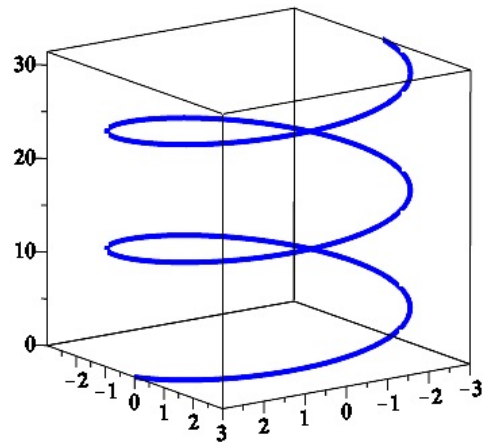
43: Determine $\int \mathbf{p}(t)dt$ for each of the following vector-valued functions:

1. $\mathbf{p}(t) = (4t^3, 3t^2, 2t, 1, 0)$. *Solution:* $(t^4, t^3, t, 0) + (C_1, C_2, C_3, C_4)$
2. $\mathbf{p}(t) = (t^2, 1 + 7t, 2/t)$. *Solution:* $(\frac{t^3}{3}, t + 7\frac{t^2}{2}, 2 \ln |t|) + (C_1, C_2, C_3)$
3. $\mathbf{p}(t) = (e^t, \tan t, \ln t)$. *Solution:* $(e^t, -\ln \cos t, t \ln t - t) + (C_1, C_2, C_3)$

Problem A: Let $\mathbf{f}(t) = (a \cos t, a \sin t, bt)$ with a and b nonzero constants. Sketch the graph of this curve (a **helix**) for $0 \leq t \leq 5\pi$. Show that the speed is constant and the velocity vector is always orthogonal to the vector $\mathbf{r}(t) = (a \cos t, a \sin t, 0)$.

Solution: The velocity vector is $\mathbf{v} = (-a \sin t, a \cos t, b)$ so speed $= |\mathbf{v}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$

and $\mathbf{v} \cdot \mathbf{r} = -a^2 \cos t \sin t + a^2 \cos t \sin t + 0t = 0$.



The Helix with $a = 3$ and $b = 2$.