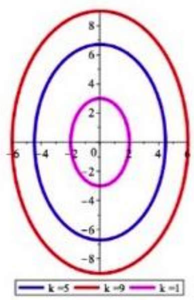
MATH 223 Spring 2022
Notes on Assignment 6

Write out careful and complete solutions of Exercise 12 (old 10) in Chapter 3 and the exercises below.

Problem 12 (Old 10)

Sketch and discuss the level curves $\frac{x^2}{4} + \frac{y^2}{9} = k$ for $k = 9, 5, 1, 0, -1$

Solution: At each level, k, the set of all points for which $f(x, y) = k$ is an ellipse centered at the origin. When $k = 0$ the level set is the origin, and when $k = -1$ the level set is empty.



1. In class, we examined the function given by $f(x, y) = \frac{xy}{x^2+2y^2}$.

(a) Show that 3/19 is a possible value for this function by exhibiting a specific point (a,b) such that $f(a, b) = \frac{3}{19}$.

Solution: Choosing a point on the line $y = mx$ yields $f(x, y) = \frac{m}{1+2m^2}$ which equals k when $2km^2 - m + k = 0$. The quadratic formula yields $m = \frac{1 \pm \sqrt{1-8k^2}}{4k}$. Setting $k = \frac{3}{19}$, yields two values m = 3 and m = 1/6. Any points on these two lines will work; for example (1,3) or (6,1)

(b) Show that there is no point (a,b) such that $f(a, b) = 1$. Here the quadratic formula with k =1 gives $m = \frac{1 \pm \sqrt{1-8}}{4}$ which has no real solutions.

(c) What is the largest possible value **M** of this function? For the quadratic formula to give real roots, we need $1 - 8k^2 \geq 0$ so $k^2 \leq \frac{1}{8}$ so largest value is $\frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

2. Let g be the function defined by $g(x, y) = \frac{xy}{2x^2+3y^2}$.

(a) What is the domain of this function? *All points except the origin*. Note: denominator will be positive.

(b) Show that $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist. *As in our class example, consider points on the line y = mx where* $g(x, mx) = \frac{m}{2+3m^2}$ *for all nonzero x. Since we get different values for different m's, there is no limit.*

(c) For which points (x,y) in the plane is $g(x,y) > 0$? *First or Third Quadrant; need xy > 0; x and y have same sign.*

(d) For which points if $g(x,y) < 0$? *Second or Fourth Quadrant*; *x and y have opposite signs so xy < 0*

(e) What is the image of this function?

Solution: The image would be all real numbers of the form $\frac{m}{2+3m^2}$. Setting this fraction equal to k and solving yields $m = \frac{1 \pm \sqrt{1-24k^2}}{6k}$ so we get real solutions exactly when $k^2 \leq \frac{1}{24}$ so $|k| \leq \frac{1}{\sqrt{24}}$. The image is the closed interval from $-\frac{1}{\sqrt{24}}$ to $\frac{1}{\sqrt{24}}$.

3. Let f be the function defined by $f(x,y) = \frac{x^2y}{x^4+y^2}$.

(a) Show that the limit of f as (x,y) approaches the origin along any line is 0.

Solution: Along vertical axis, $f(x,y) = f(0,y) = \frac{0y}{0+y^2} = 0$ for all $y \neq 0$ so limit is 0.

Similarly, along horizontal axis, $f(x,y) = f(x,0) = \frac{x^2 0}{x^4+0} = 0$ for all $x \neq 0$ so limit is 0.

Along any other line $y=mx$, $f(x,y) = f(x,mx) = \frac{x^2mx}{x^4+m^2x^2} = \frac{mx}{x^2+m^2}$ which approaches $\frac{0}{m^2} = 0$.

(b) Show that the limit of f as (x,y) approaches the origin along the curve $y = x^2$ is ½.

Solution: $f(x,y) = f(x,x^2) = \frac{x^2x^2}{x^4+x^4} = \frac{x^4}{2x^4} = \frac{1}{2}$ for all $x \neq 0$ so limit is $\frac{1}{2}$.

(c) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

Solution: No, the limit does not exist because we get different predictions coming along different paths.