MATH 223: Multivariable Calculus

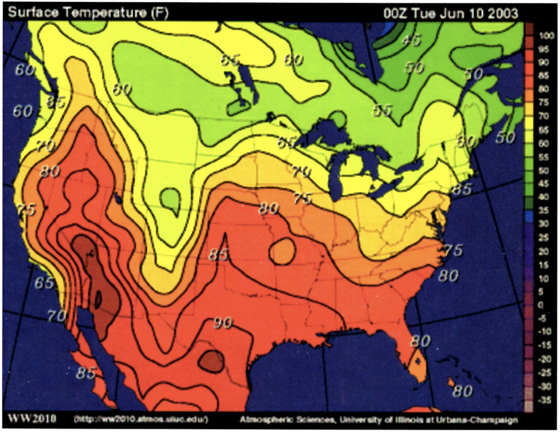
Class 6
September 22, 2023

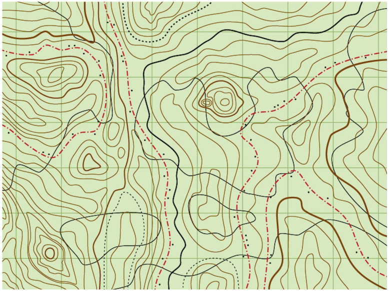
Level Curves

Given: output k (some constant)

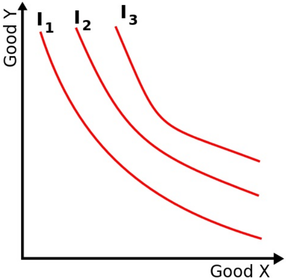
Find: All inputs Which Produce That Output

Examples Isotherms, Isobars, Isoclines, Indifference Curves





Topographic Map



Indifference Curve: Level Curve for Utility Function

FUNDAMENTAL DIFFERENCE BETWEEN DOMAIN BEING SUBSET OF $\mathcal{R}^1$ AND SUBSET OF $\mathcal{R}^n, n > 1$:

Implications for Continuity, Derivative, Integral
These all depend on **LIMIT**:

In $\mathcal{R}^1 : \lim_{x \to a} f(x)$: 2 ways to approach a

In $\mathcal{R}^n : \lim_{\mathbf{x} \to \mathbf{a}} f(x)$:infinitely many ways to approach $\mathbf{a}$

Example:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Approach along line $y = mx$:

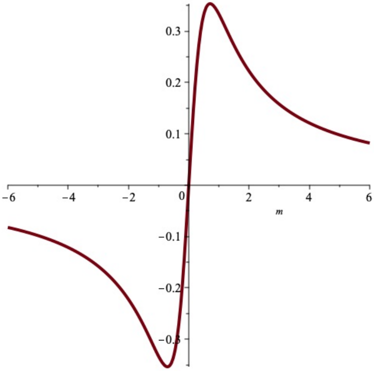
$$f(x,mx) = \frac{x(mx)}{x^2+2(mx)^2} = \frac{mx^2}{x^2+2m^2x^2} = \frac{mx^2}{x^2(1+2m^2)} = \frac{m}{1+2m^2}$$

m	$\frac{m}{1+2m^2}$
1	$\frac{1}{1+2} = \frac{1}{3}$
2	$\frac{2}{1+8} = \frac{2}{9}$
-1	$\frac{-1}{1+2} = -\frac{1}{3}$

What are the possible values of $g(m) = \frac{m}{1+2m^2}$?

$g'(m) = \frac{1-2m^2}{(1+2m^2)^2}$ and $g''(m) = \frac{4m(2m^2-3)}{(1+2m^2)^3}$

- $g(m) \to 0$ as $m \to \pm\infty$
- $g'(m) > 0$ for $\sqrt{2}/2 \leq m \leq \sqrt{2}/2$
- Point of Inflection at $\left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$



Derivative

$$f : \mathcal{R}^1 \to \mathcal{R}^1 : f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f : \mathcal{R}^1 \to \mathcal{R}^m : \mathbf{f}'(x) = \lim_{h \to 0} \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h}$$

What about $f : \mathcal{R}^n \to \mathcal{R}^1$?

$$f'(\mathbf{x}) \;\; \textbf{?=?} \;\; \lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})}{\mathbf{h}}$$

Major Problems

Division by vector **h** makes no sense.

Infinitely many ways $\mathbf{h} \to \mathbf{0}$.

Partial Solution

Consider 2 Special Ways for $\mathbf{h}$

$\mathbf{h} = (t, 0)$	$\mathbf{h} = (0, t)$
$lim_{t\to 0} \frac{f(x+t,y)-f(x,y)}{t}$	$lim_{t\to 0} \frac{f(x,y+t)-f(x,y)}{t}$
$\partial f/\partial x, f_x, D[1](f)$	$\partial f/\partial y, f_y, D[2](f)$
Partial Derivative	Partial Derivative
With Respect to x	With Respect to y
Treat y as a constant	Treat x as a constant
Use usual rules	Use usual rules
of differentiation on x	of differentiation on y

Example: $f(x,y) = x^2 y$ at point (3,4)

$f_x(x,y) = 2xy$	$f_y(x,y) = x^2$
$f_x(3,4) = 2 \times 3 \times 4 = 24$	$f_y(3,4) = 3^2 = 9$

$$
\begin{aligned}
f_x(x,y) &= \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t} \\
&= \lim_{t\to 0} \frac{(x+t)^2 y - x^2 y}{t} \\
&= \lim_{t\to 0} \frac{(x^2 + 2xt + t^2)y - x^2 y}{t} \\
&= \lim_{t\to 0} \frac{x^2 y + 2xyt + t^2 y - x^2 y}{t} \\
&= \lim_{t\to 0} \frac{2xyt + t^2 y}{t} \\
&= \lim_{t\to 0} (2xy + ty) \\
&= 2xy
\end{aligned}
$$

Example: $f(x, y) = x^2 y$

$$
\begin{aligned}
f_y(x, y) &= \lim_{t \to 0} \frac{f(x, y+t) - f(x, y)}{t} \\
&= \lim_{t \to 0} \frac{x^2(y+t) - x^2 y}{t} \\
&= \lim_{t \to 0} \frac{x^2 y + +x^2 t - x^2 y}{t} \\
&= \lim_{t \to 0} \frac{x^2 t}{t} \\
&= \lim_{t \to 0} x^2 \\
&= x^2
\end{aligned}
$$