Class 5

MATH 223 Multivariable Calculus

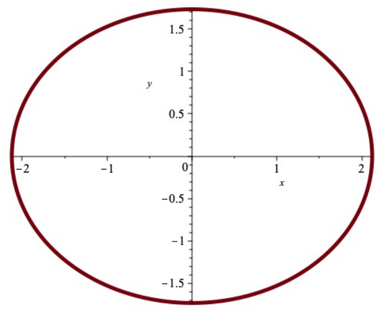
September 20, 2023

Today: Begin Examination of $f : \mathcal{R}^n \to \mathcal{R}^1$
Real-Valued Functions of Vectors
(I) Defining and Using Derivative
Chapters 3 - 5
(II) Integral
Chapters 6 - 7

Initial Focus: Geometry of Such Functions
Some Useful Pictures (A)f $f : \mathcal{R}^2 \to .\mathcal{R}^1 \quad z = f(x, y)$
Image: Interval of Real Numbers
Domain: Region in Plane
Graph: Surface in $\mathcal{R}^3$
In General, grapg $f : \mathcal{R}^n \to .\mathcal{R}^1$ is the set of points of the form $(\mathbf{x}, f(\mathbf{x}))$, an n-dimensional surface ib $n + 1$-dimensional space.
(B) Contours Level Curves for $f : \mathcal{R}^2 \to .\mathcal{R}^1$
Level Surfaces for $f : \mathcal{R}^3 \to .\mathcal{R}^1$
(C) Cross-Sections
Fix one variable
Example: $f(x, y) = 2x^2 + 3y^2$ or $z = 2x^2 + 3y^2$.
Observations:

1. $z \geq 0$ for all x, y; $z =$ only at (0,0)
2. Hold z fixed, say. $z = z_0 > 0$

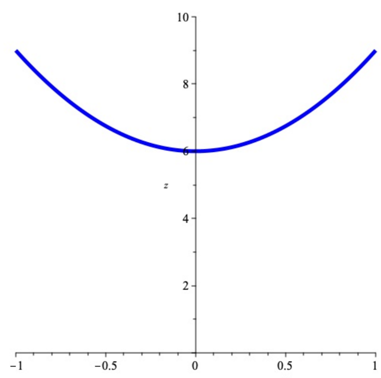
$$2x^2 + 3y^2 = z_0$$



3. Hold x fixed: $x = x_0$

$$2x_0^2 + 3y^2 = z$$

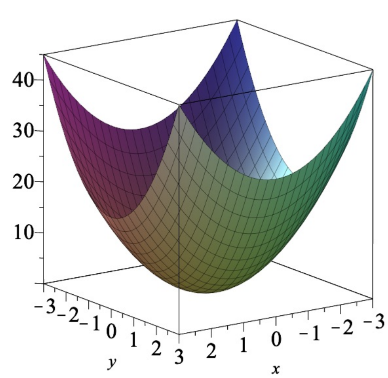
Parabola in (y, z)-plane



4. Hold y fixed: $y = y_0$

$$2x^2 + 3y_0^2 = z$$

Parabola in (x, z)-plane



Level Curves

Given: output k (some constant)

Find: All inputs Which Produce That Output
Examples Isotherms
Isobars
Isoclines
Indifference Curves

FUNDAMENTAL DIFFERENCE BETWEEN DOMAIN BEING SUBSET OF $\mathcal{R}^1$ AND SUBSET OF $\mathcal{R}^n, n > 1$:

Implications for Continuity, Derivative, Integral
These all depend on **LIMIT**:

$$\lim_{x \to a} f(x) \text{ in } \mathcal{R}^1 : \text{ 2 ways to approach } a$$

$$\lim_{\mathbf{x} \to \mathbf{a}} f(x) \text{ in } \mathcal{R}^n : \text{Infinitely Many Ways to approach } \mathbf{a}$$

Example:

$$f(x,y) = \{ \begin{matrix} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{matrix}$$

Approach along line $y = mx$:

$$f(x, mx) = \frac{x(mx)}{x^2 + 2(mx)^2} = \frac{mx^2}{x^2 + 2m^2x^2} = \frac{mx^2}{x^2(1+2m^2)} = \frac{m}{1+2m^2}$$

m	$\frac{m}{1+2m^2}$
1	$\frac{1}{1+2} = \frac{1}{3}$
2	$\frac{2}{1+8} = \frac{2}{9}$
-1	$\frac{-1}{1+2} = -\frac{1}{3}$

What are the possible values of $g(m) = \frac{m}{1+2m^2}$?
$g'(m) = \frac{1-2m^2}{(1+2m^2)^2}$ and $g''(m) = \frac{4m(2m^2-3)}{(1+2m^2)^3}$

- $g(m) \to 0$ as $m \to \pm\infty$
- $g'(m) > 0$ for $\sqrt{2}/2 \leq m \leq \sqrt{2}/2$

- Point of Inflection at $\left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$