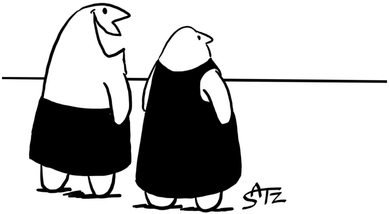
MATH 223: Multivariable Calculus



"I believe we live on the inside surface of a huge hollow sphere. It just looks flat because the curvature is so large."

Class 30: November 27, 2023


WELCOME
BACK

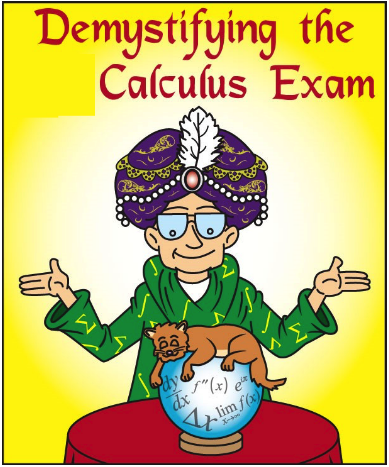
Handouts

Notes on Assignment 28

Assignment 29

Normal Vectors and Curvature

Flow Lines, Divergence and Curl



Exam 3: Wednesday Night at 7 PM

You May Bring One Sheet (Two-Sided) of Notes

Announcements

Chapter 7: Integrals and Derivatives on Curves
Chapter 8: Vector Field Theory

Today: Curvature
Introduction to Flow Lines and Divergence

Wednesday: Divergence and Curl
Friday: Conservative Vector Fields
Monday: Green's Theorem in the Plane

Normal Vectors and Curvature

Goal: Derive a Measure of Shape of a Curve.

How "Curvy" is a Curve?

Setting: Curve γ lies in $\mathbb{R}^2$ or $\mathbb{R}^3$

Parametrization $\mathbf{g}$ whose image is γ.

Some texts use $\mathbf{r}$ or $\mathbf{x} = \mathbf{x}(t)$ for the parametrization

Arc Length traversed by time t is denoted $s(t)$ and is a scalar quantity with

$$s(t) = \int |\mathbf{g}'(t)| \, dt$$

Arc Length is Integral of Speed

Speed is Derivative of Arc Length:

$$s'(t) = |\mathbf{g}'(t)|$$

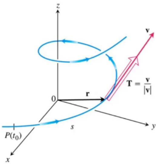
so we will have $\mathbf{g}'(t) = s'(t)\mathbf{T}(t)$

where **T** is unit tangent vector $\frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$

Unit Tangent Vector

The **unit tangent vector** gets its own notation:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$



$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$$

Example $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2+b^2}$

$$\mathbf{T}(t) = \frac{g'(t)}{|g'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

Principal Normal Vector

Start With Observation: $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$

Now differentiate both sides with respect to t:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 2\mathbf{T} \cdot \mathbf{T}' = 0$$

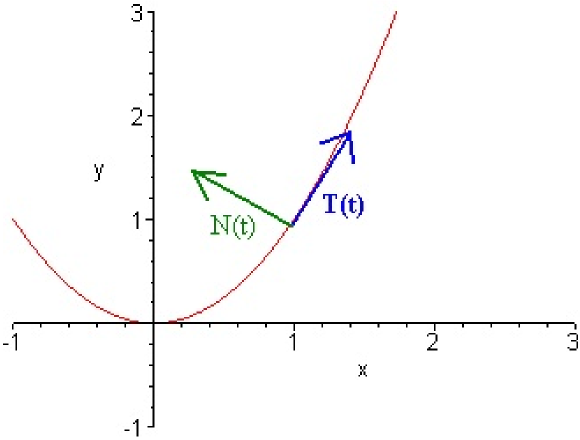
So $\mathbf{T} \cdot \mathbf{T}' = 0$

The vectors $\mathbf{T}$ and $\mathbf{T}'$ are Orthogonal

The Principal Normal Vector

$$\eta(t) = \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

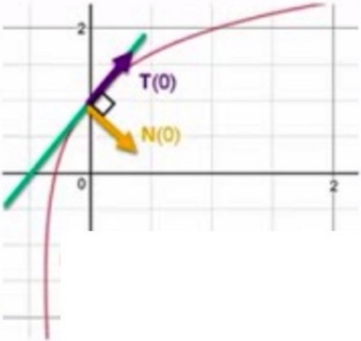
Sometimes written as $\mathbf{N} = \frac{\mathbf{T'}}{|\mathbf{T'}|}$ or $\mathbf{n} = \frac{\mathbf{t}}{|\mathbf{t}|}$



3
2
1
-1
y
N(t)
T(t)
-1
1
2
3
x

Principal Unit Normal Vector

- $0 = 1' = (T \cdot T)'$
  $= T' \cdot T + T \cdot T'$
  $= 2T \cdot T'$



- $N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

Principal Normal

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

Example $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2 + b^2}$

$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

$$\mathbf{N} = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}} \times \frac{\sqrt{a^2+b^2}}{a} = \frac{(-a\cos t, -a\sin t, 0)}{a}$$

$$\mathbf{N} = (-\cos t, -\sin t, 0)$$

$$\mathbf{N}\cdot\mathbf{T} = \frac{a\sin t\cos t - a\sin t\cos t + 0}{\sqrt{a^2+b^2}} = 0.$$

Example: Parabola in the Plane

$$\mathbf{g}(t) = (t, t^2)$$

$$\mathbf{g}'(t) = (1, 2t)$$

$$|\mathbf{g}'(t)| = \sqrt{1+4t^2}$$

$$\mathbf{T} = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(1,2t)}{\sqrt{1+4t^2}} = ((1+4t^2)^{-1/2}, 2t(1+4t^2)^{-1/2})$$

Differentiating with respect to t and simplifying, we get

$$\mathbf{T}' = \left(\frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$$

After some algebra, $|\mathbf{T}'| = \frac{2}{1+4t^2}$

$$\mathbf{N} = \left(\frac{-2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}},\right)$$

Check that $\mathbf{N} \cdot \mathbf{T} = 0$

Curvature

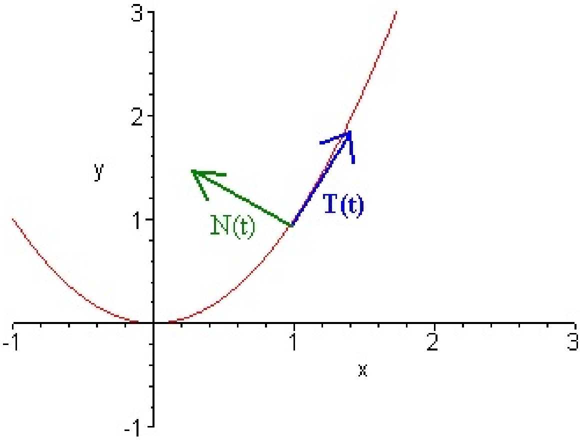
Recall $s'(t) = |\mathbf{g}'(t)|$ or, more compactly, $s' = |\mathbf{g}'|$
and $\mathbf{T} = \frac{\mathbf{g}'}{|\mathbf{g}'|} = \frac{\mathbf{g}'}{s'}$ we have $\mathbf{g}' = s'\mathbf{T}$.
Differentiate with respect to t:

$$\mathbf{g}'' = \mathbf{g}'' = (s'\mathbf{T})' = s''\mathbf{T} + s'\mathbf{T}'$$

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\mathbf{T}'$
acceleration vector		component in direction of $\mathbf{T}$		component in direction of $\mathbf{T}'$

Replace $\mathbf{T}'$ by $|\mathbf{T}'|\mathbf{N}$:

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'|\mathbf{T}'|\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration



3
2
1
-1
y
x
-1
1
2
3
N(t)
T(t)

Curvature

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\lvert\mathbf{T}'\rvert\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration

Curvature is a measure of the bend

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right|$$

<u>Theorem</u>: $\kappa = \frac{|\mathbf{T}'|}{s'} = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}$.

Proof: $\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\mathbf{T}'}{s'}$

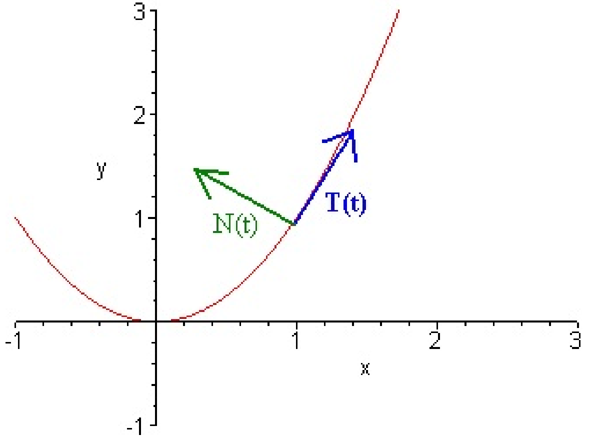
$$\boxed{\kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}}$$

$$\text{Curvature: } \kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}$$

Example Our Parabola $\mathbf{g}(t) = (t, t^2)$

We found $|\mathbf{T}'| = \frac{2}{1+4t^2}$ and $|\mathbf{g}'(t)| = \sqrt{1+4t^2}$

Thus Curvature $= \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|} = \frac{2}{1+4t^2} \times \frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}$



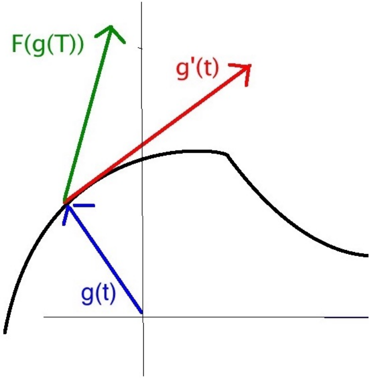
Flow Lines

Suppose γ is a curve in $\mathbb{R}^n$ which has a parametrization g.

At each point on the curve, we can associate two vectors:

Tangent Vector: $\mathbf{g'}(t)$

Vector Field: $\mathbf{F}(\mathbf{g}(t))$



If the two vectors coincide, then γ is called a **flow line** for **F**.

Hard Problem: Given $\mathbf{F}$, find flow lines
(Central Question in Differential Equations)

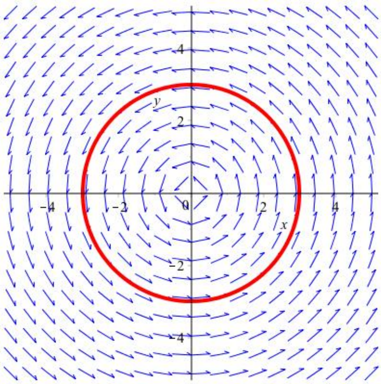
Easy Problem: Given $\mathbf{g}$ and $\mathbf{F}$, check if γ is a flow line for $\mathbf{F}$.

Example: $\mathbf{g}(t) = (3\cos\frac{t}{12}, 3\sin\frac{t}{12})$

Then $\mathbf{g'}(t) = (-\frac{1}{4}\sin\frac{t}{12}, \frac{1}{4}\cos\frac{t}{12})$

Suppose $\mathbf{F}(x,y) = \left(\frac{-y}{4\sqrt{x^2+y^2}}, \frac{x}{4\sqrt{x^2+y^2}}\right)$

Then $\mathbf{F}(x,y) = \left(\frac{-3\sin\frac{t}{12}}{4\times 3}, \frac{3\cos\frac{t}{12}}{4\times 3}\right) = \mathbf{g'}(t)$



Flow Lines and Differential Equations

Star with a system of differential equations

$$\frac{dx}{dt} = (2-y)(x-y) = f(x,y)$$
$$\frac{dy}{dt} = (1+x)(x+y) = g(x,y)$$

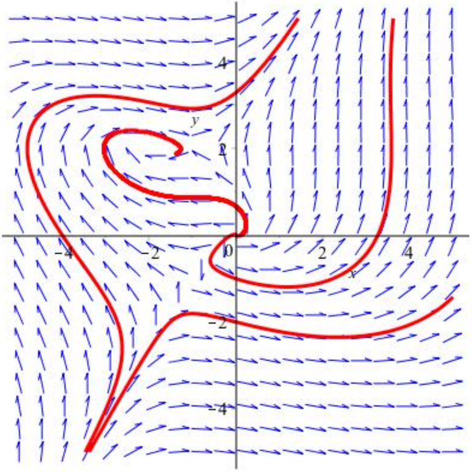
Can write as a single equation:

$$\frac{dy}{dx} = \frac{(1+x)(x-y)}{(2-y)(x-y)} = \frac{g(x,y)}{f(x,y)}$$

Observe:

1. Solution of the equation is a curve in the (x,y)-plane
2. As time goes forward, point moves along the curve in accordance to the equation
3. $\mathbf{F}(x,y) = (f(x,y), g(x,y))$ is a vector field.
4. At each point on curve, direction of motion is given by the vector field
5. The vector field is tangent to the curve
6. The curve is tangent to the vector field

Definition:A **flow line** of a vector field **F** is a differentiable function **g** such that the velocity vector **g'** at each point coincides with the field vector **F(g)**.



Divergence of a Vector Field

Definition div $\mathbf{F}$ = trace of $\mathbf{F}'$ of the Jacobi Matrix

Example **F**: $\mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (2x - y, x - 3y)$

$$\mathbf{F} = \begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix} \text{ implies div } \mathbf{F} = 2 - 3 = -1$$

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x, y, z) = (xy, yz, zx)$

$$\mathbf{F}' = \begin{pmatrix} y & -- & -- \\ -- & z & -- \\ -- & -- & x \end{pmatrix} \text{ implies div } \mathbf{F} = y + z + x$$

Example: $\mathbf{F}$: $\mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{F}(x, y, z) = (yz, xz, xy)$

Alternate Notation: $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$\mathbf{F'} = \begin{pmatrix} 0 & z & y \\ z & 0 & y \\ y & z & 0 \end{pmatrix} \text{ implies div } \mathbf{F} = 0$$

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

$$\mathbf{F'} = \begin{matrix} F1 \\ F2 \\ F3 \end{matrix} \begin{pmatrix} 0 & z & y \\ z & 0 & y \\ y & z & 0 \end{pmatrix} \text{ implies div } \mathbf{F} = 0$$

In general, **div F is a real -valued function of n variables**.

Notes

1. Gauss's Theorem: $\int_R \text{ div } \mathbf{F}\, dV = \int_{\partial R} \mathbf{F} \cdot d\mathbf{S}$

2. div $\mathbf{F}$ gives expansion rate of fluid at point $\mathbf{x}$
   div $\mathbf{F} > 0$ means fluid is expanding, getting less dense
   div $\mathbf{F} < 0$ means fluid is contracting, becomes more dense

3. Alternate Notation; $\mathbf{F} = (F_1, F_2, F_3), \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
   Then div $\mathbf{F} = \mathbf{F} \cdot \nabla$

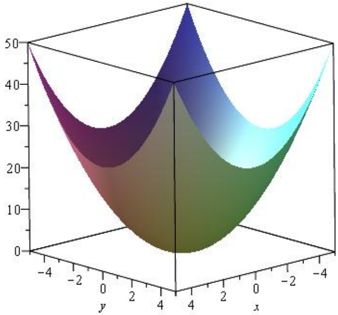
Example

$$\mathbf{F}(x,y,z) = (xy^2 + z\ln(1+y^2), \sin(xz) - zy, x^2z + \arctan y + e^{x^2})$$

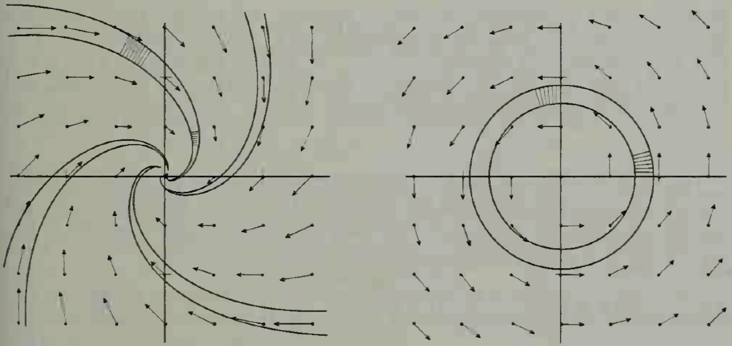
$$\text{div } \mathbf{F} = y^2 - z + x^2$$

so div $\mathbf{F} > 0$ if $x^2 + y^2 > z$

$z = x^2 + y^2$ is equation of elliptic paraboloid.



Divergence is positive on the outside, negative on the inside.



$\mathbf{F}(x, y) = \frac{1}{8}(-x + y)\mathbf{i} + \frac{1}{8}(-x - y)\mathbf{j}$
Area decreased: $\mathrm{div}\,\mathbf{F}(x, y) = -\frac{1}{4}$

(a)

$\mathbf{G}(x, y) = -\frac{1}{4}(y/\sqrt{x^2 + y^2})\mathbf{i} + \frac{1}{4}(x/\sqrt{x^2 + y^2})\mathbf{j}$
Area preserved: $\mathrm{div}\,\mathbf{G}(x, y) = 0$

(b)

Figure 8.10