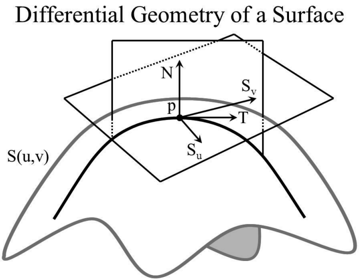
MATH 223: Multivariable Calculus



Class 29: November 17, 2023

Notes on Assignment 27

Assignment 28

Normal Vectors and Curvature

"Sit and stay were no problem but she's hit a wall with multivariable calculus."

Exam 3: Wednesday Night at 7 PM

You May Bring One Sheet (Two-Sided) of Notes

Announcements

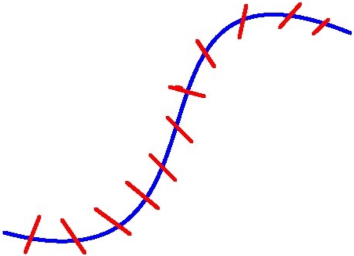
Chapter 7: Integrals and Derivatives on Curves

Today: Weighted Curves and Surfaces of Revolution
Conservation of Energy
Normal Vectors and Curvature

After Thanksgiving: Monday: Flow Lines, Divergence and Curl
Wednesday: Conservative Vector Fields

Mass of a Weighted Curve

Density (μ) is mass per unit length



Total Mass $\sim \sum \mu(point) \times$ Length of short piece of curve

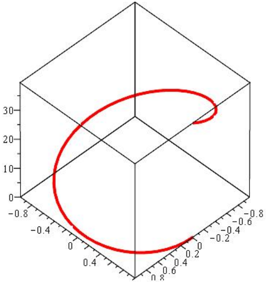
Total **Mass** $= \int \mu(g(t))|g'(t)|\, dt$

Total **Mass** : $\int \mu(g(t))|g'(t)|\,dt$

<u>Example</u> Spacecurve $g(t) = (\sin t, \cos t, t^2), 0 \le t \le 2\pi$

Here $g'(t) = (\cos t, -\sin t, 2t)$

so $|g'(t)| = \sqrt{\cos^2 t + \sin^2 t + 4t^2} = \sqrt{1+4t^2}$



Suppose $\mu(x,y,z) = x^2 + y^2 + \sqrt{z} - 1$

Then $\mu(g(t)) = \mu(\sin t, \cos t, t^2) = \cos^2 + \sin^2 t + \sqrt{t^2} - 1$

$= 1 + t - 1 = t$

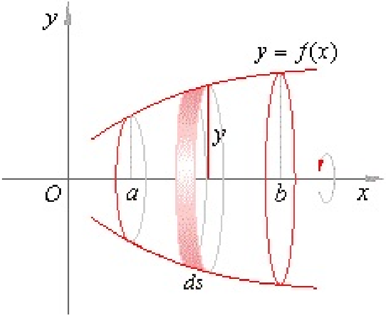
Thus **Mass** $= \int_0^{2\pi} t\sqrt{1+4t^2}\,dt$

$$= \frac{1}{12}(1+4t^2)^{3/2}\Big|_0^{2\pi} = \frac{1}{12}\left[(1+16\pi^2)^{3/2} - 1\right]$$

Surface of Revolution

S is a surface in $\mathbb{R}^3$ obtained by rotating a plane curve about a straight line in the plane.

Simplest Case: Rotate $y = f(x)$ about x-axis.



$$\text{Volume } = \int_a^b \pi\,[f(x)]^2\;dx$$

$$\text{Surface Area } = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

Volume $= \int_a^b \pi \left[f(x)\right]^2 \, dx$

Surface Area $= \int_a^b 2\pi f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx$

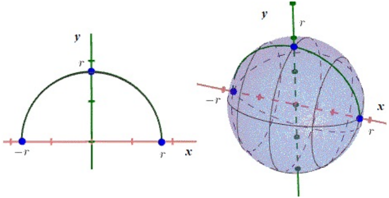
Suppose curve has parametrization $g : \mathbb{R}^1 \to \mathbb{R}^2, t_0 \le t \le t_1$

$g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.

Volume $= \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t)\, dt$

Surface Area $= \int_{t_0}^{t_1} 2\pi y(t)|g'(t)|\, dt$

Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r\cos t, r\sin t), 0 \leq t \leq \pi$$

$$\text{Volume } = \int_{t_0}^{t_1} \pi\, [y(t)]^2\, x'(t)\, dt$$

$$\text{Surface Area } = \int_{t_0}^{t_1} 2\pi y(t) |g'(t)|\, dt$$

$$\text{Surface Area } = \int_{t_0}^{\pi} r^2 2\pi \sin t\, dt$$

$$= -2\pi r^2 \cos t\Big|_0^{\pi} = -2r^2\pi(-1-1) = 4\pi r^2.$$

$$\text{Volume } = \int_0^{\pi} \pi(r\sin t)^2 r \sin t\, dt = \frac{4}{3}\pi r^3$$

If **F** $= \nabla f$ for some f, then we call **F** a **Conservative Vector Field** or an **Exact Vector Field**

and f is called a **Potential** of **F**

The function $P(\vec{x}) = -f(\vec{x})$ is the **Potential Energy** of the field **F**.

Conservative Vector Field: $\mathbf{F}(x, y) = (2xy, x^2 + 2y)$
Nonconservative Example $\mathbf{F}(x, y) = (x, x + 1)$

Application: **Conservation of Energy**

Suppose $\mathbf{g}(t)$ represents the position of an object of varying mass $m(t)$ in space at time t.

The velocity vector of the object is $\mathbf{v} = \mathbf{g}'(t)$.

The Force acting on the object at position $g(t)$ is

$$\mathbf{F}(\mathbf{g}(t)) = [m(t)\mathbf{v}(t)]' = m'(t)\mathbf{v}(t) + m(t)\mathbf{v}'(t)$$

Then

$$\begin{aligned}
\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) &= \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{v}(t) \\
&= \left[m'(t)\mathbf{v}(t) + m(t)\mathbf{v}'(t)\right] \cdot \mathbf{v}(t) \\
&= m'(t)\mathbf{v}(t) \cdot \mathbf{v}(t) + m(t)\mathbf{v}'(t) \cdot \mathbf{v}(t) \\
&= m'(t)s^2(t) + m(t)s'(t)s(t)
\end{aligned}$$

where $s(t) = |\mathbf{v}(t)| =$ **speed** at time t.

To Show: $s'(t)s(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t)$

Start with $s^2(t) = |\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$

Differentiate each side with respect to t:

$$2s(t)s'(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}'(t) \cdot \mathbf{v}(t)$$

Thus $s'(t)s(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t)$
and

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)s^2(t) + m(t)s'(t)s(t)$$

Application: Conservation of Energy

(a) $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)s^2(t) + m(t)s'(t)s(t)$

We'll use the scalar v for the scalar s

so $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)v^2(t) + m(t)v'(t)v(t)$

(b) $m(t) =$ Constant implies $m' = 0$

so $\mathbf{F}(g(t)) \cdot g'(t) = mv(t)v'(t)$

$$\int_a^b mv(t)v'(t)\,dt = \left.\frac{mv(t)^2}{2}\right|_{t=a}^{t=b}$$

Application: **Conservation of Energy**

Suppose **F** is a force field which moves an object of mass m from $\vec{a}$ to $\vec{b}$ along curve γ.

Let g be a parametrization of curve γ and $v(t) = g'(t)$.

Then the work done in moving the object is

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2$$ (**Change in Kinetic Energy)**

If **F** is a conservative field, then we can also compute work done by $\int_\gamma \mathbf{F} = f(\vec{b}) - f(\vec{a}) = p(\vec{a}) - p(\vec{b}) =$ **Change in Potential Energy**

Equating the two expressions for work, we have

$$\tfrac{1}{2}m|v(t_b)|^2 - \tfrac{1}{2}m|v(t_a)|^2 = p(\vec{a}) - p(\vec{b})$$

$$p(\vec{b}) + \tfrac{1}{2}m|v(t_b)|^2 = p(\vec{a}) + \tfrac{1}{2}m|v(t_a)|^2$$

where $\vec{a}$ and $\vec{b}$ are any 2 points

So Sum of Potential and Kinetic Energy is Constant

Law of Conservation of Total Energy

Normal Vectors and Curvature

Goal: Derive a Measure of Shape of a Curve.

How "Curvy" is a Curve?

Setting: Curve γ lies in $\mathbb{R}^2$ or $\mathbb{R}^3$

Parametrization $\mathbf{g}$ whose image is γ.

Some texts use $\mathbf{r}$ or $\mathbf{x} = \mathbf{x}(t)$ for the parametrization

Arc Length traversed by time t is denoted $s(t)$ and is a scalar quantity with

$$s(t) = \int |\mathbf{g}'(t)| \, dt$$

Arc Length is Integral of Speed

Speed is Derivative of Arc Length:

$$s'(t) = |\mathbf{g}'(t)|$$

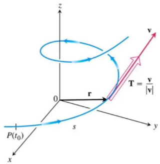
so we will have $\mathbf{g}'(t) = s'(t)\mathbf{T}(t)$

where **T** is unit tangent vector $\frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$

Unit Tangent Vector

The **unit tangent vector** gets its own notation:

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$



$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}$$

Example $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2+b^2}$

$$\mathbf{T}(t) = \frac{g'(t)}{|g'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

Principal Normal Vector

Start With Observation: $\mathbf{T} \cdot \mathbf{T} = |\mathbf{T}|^2 = 1$

Now differentiate both sides with respect to t:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 2\mathbf{T} \cdot \mathbf{T}' = 0$$

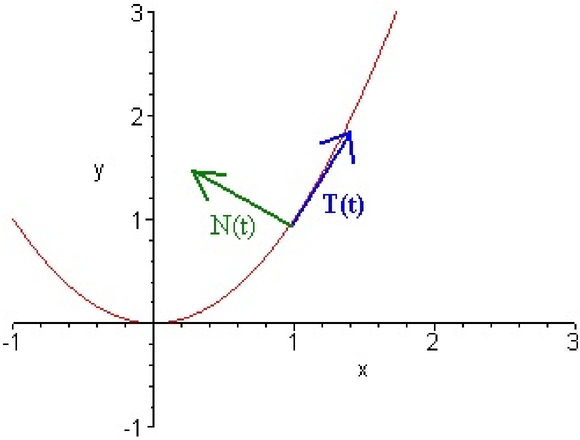
So $\mathbf{T} \cdot \mathbf{T}' = 0$

The vectors $\mathbf{T}$ and $\mathbf{T}'$ are Orthogonal

The Principal Normal Vector

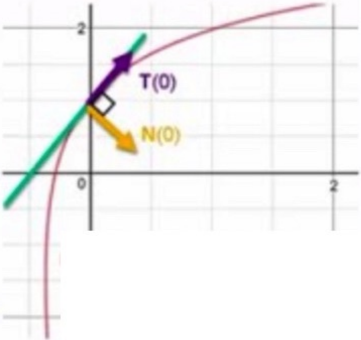
$$\eta(t) = \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

Sometimes written as $\mathbf{N} = \frac{\mathbf{T'}}{|\mathbf{T'}|}$ or $\mathbf{n} = \frac{\mathbf{t}}{|\mathbf{t}|}$


3
2
1
-1
y
N(t)
T(t)
-1
1
2
3
x

Principal Unit Normal Vector

- $0 = 1' = (\boldsymbol{T} \cdot \boldsymbol{T})'$
  $= \boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}'$
  $= 2\boldsymbol{T} \cdot \boldsymbol{T}'$



- $$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$$

Principal Normal

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

Example $\mathbf{g}(t) = (a\cos t, a\sin t, bt)$

Then $\mathbf{g}'(t) = (-a\sin t, a\cos t, b)$ and $|\mathbf{g}'(t)| = \sqrt{a^2+b^2}$

$$\mathbf{T}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

Then $\mathbf{T}' = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}}$ and $|\mathbf{T}'| = \frac{a}{\sqrt{a^2+b^2}}$

$$\mathbf{N} = \frac{(-a\cos t, -a\sin t, 0)}{\sqrt{a^2+b^2}} \times \frac{\sqrt{a^2+b^2}}{a} = \frac{(-a\cos t, -a\sin t, 0)}{a}$$

$$\mathbf{N} = (-\cos t, -\sin t, 0)$$

$$\mathbf{N}\cdot\mathbf{T} = \frac{a\sin t\cos t - a\sin t\cos t + 0}{\sqrt{a^2+b^2}} = 0.$$

Example: Parabola in the Plane

$$\mathbf{g}(t) = (t, t^2)$$

$$\mathbf{g}'(t) = (1, 2t)$$

$$|\mathbf{g}'(t)| = \sqrt{1 + 4t^2}$$

$$\mathbf{T} = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|} = \frac{(1,2t)}{\sqrt{1+4t^2}} = ((1 + 4t^2)^{-1/2}, 2t(1 + 4t^2)^{-1/2})$$

Differentiating with respect to t and simplifying, we get

$$\mathbf{T}' = \left(\frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$$

After some algebra, $|\mathbf{T}'| = \frac{2}{1+4t^2}$

$$\mathbf{N} = \left(\frac{-2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}},\right)$$

Check that $\mathbf{N} \cdot \mathbf{T} = 0$

Curvature

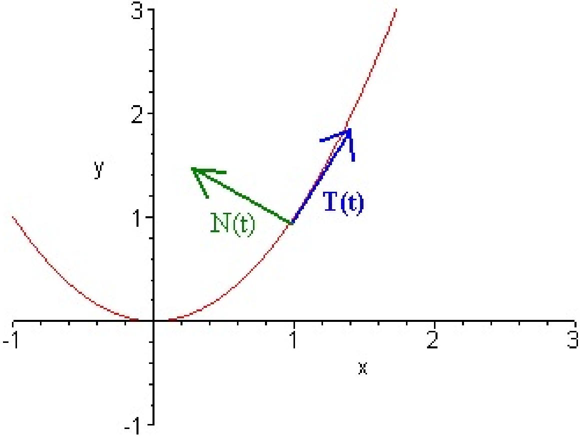
Recall $s'(t) = |\mathbf{g}'(t)|$ or, more compactly, $s' = |\mathbf{g}'|$
and $\mathbf{T} = \frac{\mathbf{g}'}{|\mathbf{g}'|} = \frac{\mathbf{g}'}{s'}$ we have $\mathbf{g}' = s'\mathbf{T}$.
Differentiate with respect to t:

$$\mathbf{g}'' = \mathbf{g}'' = (s'\mathbf{T})' = s''\mathbf{T} + s'\mathbf{T}'$$

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\mathbf{T}'$
acceleration vector		component in direction of $\mathbf{T}$		component in direction of $\mathbf{T}'$

Replace $\mathbf{T}'$ by $|\mathbf{T}'|\mathbf{N}$:

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'|\mathbf{T}'|\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration


3
2
1
-1
y
-1
1
2
3
x
N(t)
T(t)

Curvature

$\mathbf{g}''$	$=$	$s''\mathbf{T}$	$+$	$s'\lvert\mathbf{T}'\rvert\mathbf{N}$
acceleration vector		tangential acceleration		centripetal acceleration

Curvature is a measure of the bend

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right|$$

<u>Theorem</u>: $\kappa = \frac{|\mathbf{T}'|}{s'} = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}$.

Proof: $\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\mathbf{T}'}{s'}$

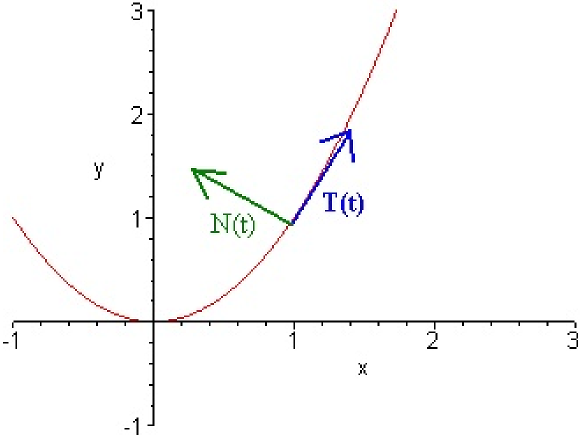
$$\boxed{\kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}}$$

$$\boxed{\text{Curvature: } \kappa = \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|}}$$

Example Our Parabola $\mathbf{g}(t) = (t, t^2)$

We found $|\mathbf{T}'| = \frac{2}{1+4t^2}$ and $|\mathbf{g}'(t)| = \sqrt{1+4t^2}$

Thus Curvature $= \frac{|\mathbf{T}'|}{|\mathbf{g}'(t)|} = \frac{2}{1+4t^2} \times \frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}$

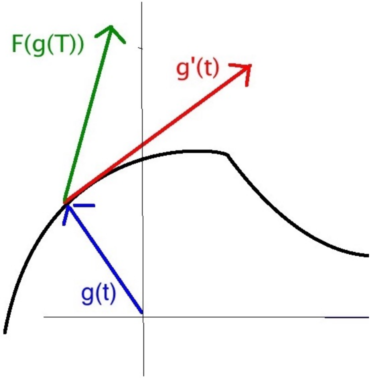


Flow Lines

Suppose γ is a curve in $\mathbb{R}^n$ which has a parametrization g. At each point on the curve, we can associate two vectors:

Tangent Vector: $\mathbf{g'}(t)$

Vector Field: $\mathbf{F}(\mathbf{g}(t))$



If the two vectors coincide, then γ is called a **flow line** for **F**.

Hard Problem: Given $\mathbf{F}$, find flow lines
(Central Question in Differential Equations)

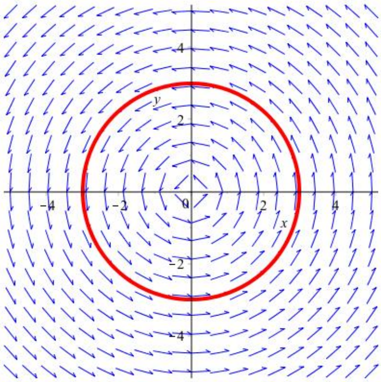
Easy Problem: Given $\mathbf{g}$ and $\mathbf{F}$, check if γ is a flow line for $\mathbf{F}$.

Example: $\mathbf{g}(t) = (3\cos\frac{t}{12}, 3\sin\frac{t}{12})$

Then $\mathbf{g}'(t) = (-\frac{1}{4}\sin\frac{t}{12}, \frac{1}{4}\cos\frac{t}{12})$

Suppose $\mathbf{F}(x, y) = \left(\frac{-y}{4\sqrt{x^2+y^2}}, \frac{x}{4\sqrt{x^2+y^2}}\right)$

Then $\mathbf{F}(x, y) = \left(\frac{-3\sin\frac{t}{12}}{4\times 3}, \frac{3\cos\frac{t}{12}}{4\times 3}\right) = \mathbf{g}'(t)$



Flow Lines and Differential Equations

Star with a system of differential equations

$$\frac{dx}{dt} = (2-y)(x-y) = f(x,y)$$
$$\frac{dy}{dt} = (1+x)(x+y) = g(x,y)$$

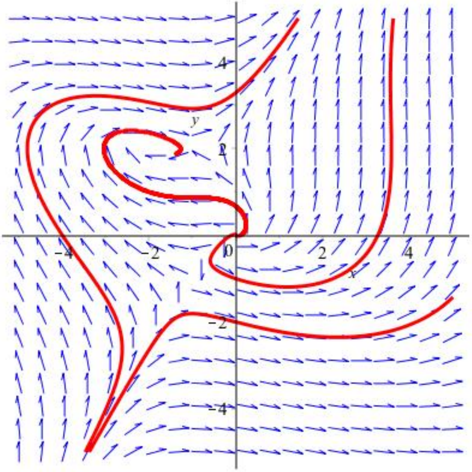
Can write as a single equation:

$$\frac{dy}{dx} = \frac{(1+x)(x-y)}{(2-y)(x-y)} = \frac{g(x,y)}{f(x,y)}$$

Observe:

1. Solution of the equation is a curve in the (x, y)-plane
2. As time goes forward, point moves along the curve in accordance to the equation
3. $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ is a vector field.
4. At each point on curve, direction of motion is given by the vector field
5. The vector field is tangent to the curve
6. The curve is tangent to the vector field

Definition:A **flow line** of a vector field **F** is a differentiable function **g** such that the velocity vector **g'** at each point coincides with the field vector **F(g)**.




May your
Thanksgiving
be full of peace, love, and joy.