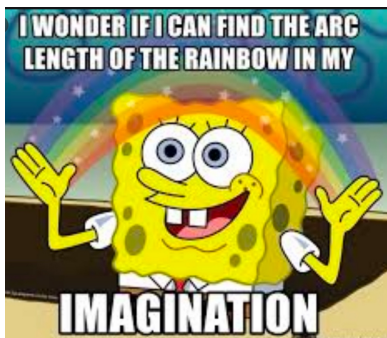


MATH 223: Multivariable Calculus



Class 28: November 15, 2023



Notes on Assignment 26
Assignment 27
Weighted Curves and Surfaces of Revolution

Team Project Update Due on Friday



Wednesday, November 29
7 PM -?

You May Bring One Sheet (Two-Sided) of Notes

Today:

Conservative Vector Fields and Conservation of Energy
Weighted Curves and Surfaces of Revolution

Friday, November 17: Normal Vectors and Curvature

Monday, November 27: Flow Lines, Divergence and Curl

INTEGRATION OF VECTOR FIELDS $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\mathbf{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$$

What is Meaning of $\int_{\mathcal{D}} \mathbf{F}$?

For Now: \mathcal{D} is a one-dimensional set in \mathbb{R}^n

\mathcal{D} is a curve parametrized by a function $g : \mathbb{R}^1 \rightarrow \mathbb{R}^n$ on an interval $a \leq t \leq b$

We denote the **image** of g by γ

Definition The **Line Integral** of \mathbf{F} over γ is

$$\int_{\gamma} \mathbf{F} \cdot d\vec{x} = \int_a^b \mathbf{F}(g(t)) \cdot g'(t) dt$$

Theorem The value of the line integral $\int_{\gamma} \mathbf{F}$ is independent of the parametrization of γ but in general is dependent on the curve itself.

For some vector fields, the line integral $\int_{\gamma} \mathbf{F}$ depends only on the **endpoints** of the curve.

In particular, this is true of \mathbf{F} is a **gradient field**; that is,

$$\mathbf{F} = \nabla f$$

for some real-valued function f .

Theorem (**The Fundamental Theorem of Calculus for Line Integrals**. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ be

continuously differentiable and let

$\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \rightarrow \mathbb{R}^n$ is a continuous curve with endpoints \vec{a} and \vec{b} .

Then $\int_{\gamma} \mathbf{F} = \int_{\gamma} \nabla f = f(\vec{b}) - f(\vec{a})$.

If $\mathbf{F} = \nabla f$ for some f , then we call \mathbf{F}
a **Conservative Vector Field**
or an **Exact Vector Field**

and f is called a **Potential** of \mathbf{F}

The function $P(\vec{x}) = -f(\vec{x})$ is the **Potential Energy** of the field
 \mathbf{F} .

Conservative Vector Field: $\mathbf{F}(x, y) = (2xy, x^2 + 2y)$

Nonconservative Example $\mathbf{F}(x, y) = (x, x + 1)$

Application: Conservation of Energy

Suppose $\mathbf{g}(t)$ represents the position of an object of varying mass $m(t)$ in space at time t .

The velocity vector of the object is $\mathbf{v} = \mathbf{g}'(t)$.

The Force acting on the object at position $\mathbf{g}(t)$ is

$$\mathbf{F}(\mathbf{g}(t)) = [m(t)\mathbf{v}(t)]' = m'(t)\mathbf{v}(t) + m(t)\mathbf{v}'(t)$$

Then

$$\begin{aligned}\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) &= \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{v}(t) \\ &= [m'(t)\mathbf{v}(t) + m(t)\mathbf{v}'(t)] \cdot \mathbf{v}(t) \\ &= m'(t)\mathbf{v}(t) \cdot \mathbf{v}(t) + m(t)\mathbf{v}'(t) \cdot \mathbf{v}(t) \\ &= m'(t)s^2(t) + m(t)s'(t)s(t)\end{aligned}$$

where $s(t) = |\mathbf{v}(t)| = \mathbf{speed}$ at time t .

To Show: $s'(t)s(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t)$

Start with $s^2(t) = |\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$

Differentiate each side with respect to t :

$$2s(t)s'(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}'(t) \cdot \mathbf{v}(t)$$

Thus $s'(t)s(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t)$

and

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)s^2(t) + m(t)s'(t)s(t)$$

Application: **Conservation of Energy**

(a) $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)s^2(t) + m(t)s'(t)s(t)$

We'll use the scalar v for the scalar s

so $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = m'(t)v^2(t) + m(t)v'(t)v(t)$

(b) $m(t) = \text{Constant}$ implies $m' = 0$

so $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = mv(t)v'(t)$

$$\int_a^b mv(t)v'(t) dt = \frac{mv(t)^2}{2} \Big|_{t=a}^{t=b}$$

Application: Conservation of Energy

Suppose \mathbf{F} is a force field which moves an object of mass m
from \vec{a} to \vec{b} along curve γ .

Let g be a parametrization of curve γ and $v(t) = g'(t)$.

Then the work done in moving the object is

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2 \quad (\text{Change in Kinetic Energy})$$

If \mathbf{F} is a conservative field, then we can also compute work done by
 $\int_{\gamma} \mathbf{F} = f(\vec{b}) - f(\vec{a}) = p(\vec{a}) - p(\vec{b}) = \text{Change in Potential Energy}$

Equating the two expressions for work, we have

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2 = p(\vec{a}) - p(\vec{b})$$

$$p(\vec{b}) + \frac{1}{2}m|v(t_b)|^2 = p(\vec{a}) + \frac{1}{2}m|v(t_a)|^2$$

where \vec{a} and \vec{b} are any 2 points

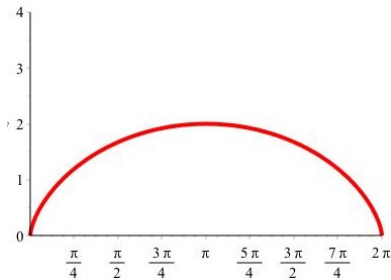
So Sum of Potential and Kinetic Energy is Constant

Law of Conservation of Total Energy

Arc Length

Let $g : \mathbb{R}^1 \rightarrow \mathbb{R}^n$ be defined on $a \leq t \leq b$. Then the image of g is a curve γ with length $L(\gamma) = \int_a^b |g'(t)| dt$.

Example: **Cycloid:** $g(t) = (t - \sin t, 1 - \cos t), 0 \leq t \leq 2\pi$



$$g'(t) = (1 - \cos t, \sin t)$$

$$|g'(t)| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2 \cos t} = \sqrt{2(1 - \cos t)} = \sqrt{2(2 \sin^2(t/2))} = 2 \sin(t/2)$$

$$L(\gamma) = \int_0^{2\pi} 2 \sin(t/2) dt = -4 \cos(t/2) \Big|_0^{2\pi} = 8$$

Other Formulations

$$L(\gamma) = \int_a^b |g'(t)| dt$$

If a curve is given by $y = f(x)$, $a \leq x \leq b$, then let $g(t) = (t, f(t))$

so

$$|g'(t)| = |(1, f'(t))| = \sqrt{1 + [f'(t)]^2}$$

If $g(t) = (h_1(t), h_2(t))$, then $|g'(t)| = \sqrt{[h_1'(t)]^2 + [h_2'(t)]^2}$.

Arc Length Parametrization

Let γ be a curve parametrized by $g(t)$ for $t_0 \leq t \leq t_1$

With $\vec{x}(t) = g(t)$, \vec{x} is position at time t .

Then **arc length function** is $s = s(t) = \int_{t_0}^t |g'(t)| dt = \int_{t_0}^t |x(t)| dt$

If $|g'(t)| = 1$ for all t , then we say the **curve is parametrized by arc length**

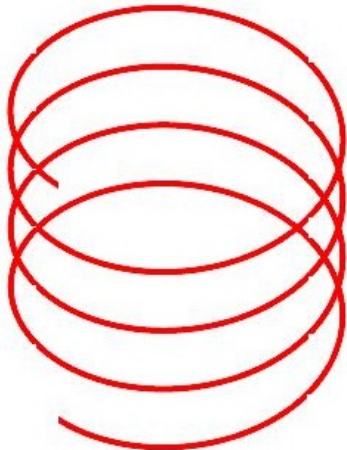
Moving along the curve with uniform speed of 1 means that at time s we are at a point s units along the curve.

Example 1: Unit Circle: $g(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$

Example 2 Helix: $g(t) = \left(\frac{a \cos t}{\sqrt{a^2+b^2}}, \frac{a \sin t}{\sqrt{a^2+b^2}}, \frac{bt}{\sqrt{a^2+b^2}} \right)$.

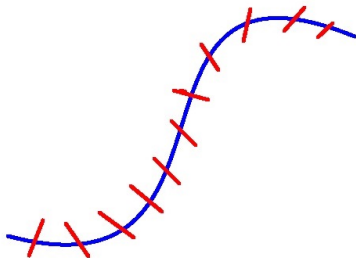
Then $g'(t) = \left(\frac{-a \sin t}{\sqrt{a^2+b^2}}, \frac{a \cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right)$.

and $|g'(t)| = \sqrt{\frac{a^2 \sin^2 t + a^2 \cos^2 t + b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1$



Mass of a Weighted Curve

Density (μ) is mass per unit length



Total Mass $\sim \sum \mu(\text{point}) \times \text{Length of short piece of curve}$

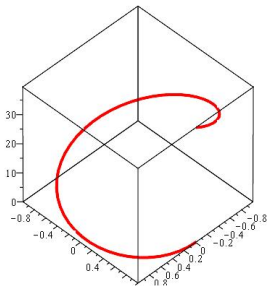
$$\text{Total Mass} = \int \mu(g(t)) |g'(t)| dt$$

Total **Mass** : $\int \mu(g(t))|g'(t)| dt$

Example Spacecurve $g(t) = (\sin t, \cos t, t^2), 0 \leq t \leq 2\pi$

Here $g'(t) = (\cos t, -\sin t, 2t)$

so $|g'(t)| = \sqrt{\cos^2 t + \sin^2 t + 4t^2} = \sqrt{1 + 4t^2}$



Suppose $\mu(x, y, z) = x^2 + y^2 + \sqrt{z} - 1$

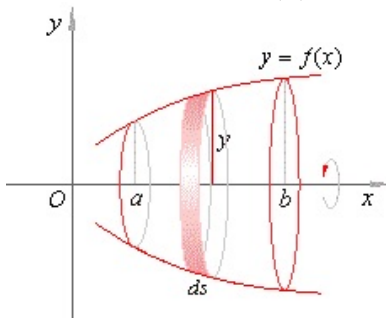
Then $\mu(g(t)) = \mu(\sin t, \cos t, t^2) = \cos^2 + \sin^2 t + \sqrt{t^2} - 1$
 $= 1 + t - 1 = t$

Thus **Mass** $= \int_0^{2\pi} t\sqrt{1 + 4t^2} dt$
 $= \frac{1}{12}(1 + 4t^2)^{3/2} \Big|_0^{2\pi} = \frac{1}{12} [(1 + 16\pi^2)^{3/2} - 1]$

Surface of Revolution

S is a surface in \mathbb{R}^3 obtained by rotating a plane curve about a straight line in the plane.

Simplest Case: Rotate $y = f(x)$ about x -axis.



$$\text{Volume} = \int_a^b \pi [f(x)]^2 dx$$

$$\text{Surface Area} = \int_a^b 2\pi \sqrt{1 + [f(x)]^2} dx$$

$$\text{Volume} = \int_a^b \pi [f(x)]^2 dx$$

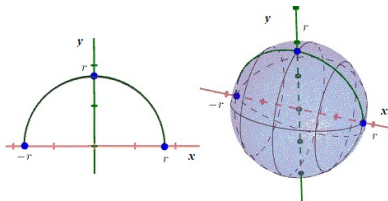
$$\text{Surface Area} = \int_a^b 2\pi \sqrt{1 + [f(x)]^2} dx$$

Suppose curve has parametrization $g : \mathbb{R}^1 \rightarrow \mathbb{R}^2, t_0 \leq t \leq t_1$
 $g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.

$$\text{Volume} = \int_{t_0}^{t_1} \pi [y(t)]^2 x'(t) dt$$

$$\text{Surface Area} = \int_{t_0}^{t_1} 2\pi y(t) |g'(t)| dt$$

Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r \cos t, r \sin t), 0 \leq t \leq \pi$$

$$\text{Volume} = \int_{t_0}^{t_1} \pi [y(t)]^2 x'(t) dt$$

$$\text{Surface Area} = \int_{t_0}^{t_1} 2\pi y(t) |g'(t)| dt$$

$$\text{Surface Area} = \int_{t_0}^{\pi} r^2 2\pi \sin t dt$$

$$= -2\pi r^2 \cos t \Big|_0^{\pi} = -2r^2\pi(-1 - 1) = 4\pi r^2.$$

$$\text{Volume} = \int_0^{\pi} \pi (r \sin t)^2 r \sin t dt = \frac{4}{3}\pi r^3$$

