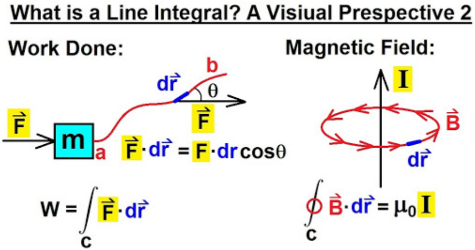
MATH 223: Multivariable Calculus



Class 27: November 13, 2023

Notes on Assignment 25

Assignment 26

Integrals and Derivatives on Curves

Announcements

Chapter 7: Integrals and Derivatives on Curves

Today: Line integral

Next Topics:

Weighted Curves and Arc Length

Surfaces of Revolution

Normal Vectors and Curvature

Integrals So Far

Real Valued Functions: $f : \mathbb{R}^n \to \mathbb{R}^1$

Iterated Integral

Multiple Integral

Vector Valued Functions

(A): $f := \mathbb{R}^1 \to \mathbb{R}^n$

$$f(\vec{t}) = (f_1(t), f_2(t), ...f_n(t))$$

so $\int_a^b f(\vec{t})\, dt = (\int_a^b f_1(t), \int_a^b f_2(t), ..., \int_a^b f_n(t)$

(B):VECTOR FIELDS **F** : $\mathbb{R}^n \to \mathbb{R}^n$

$$\mathbf{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), .., F_n(\vec{x}))$$

What is Meaning of $\int_{\mathcal{D}} \mathbf{F}$?

Today: $\mathcal{D}$ is a one-dimensional set in $\mathbb{R}^n$

$\mathcal{D}$ is a curve defined by a function $g : \mathbb{R}^1 \to \mathbb{R}^n$ on an interval $a \leq t \leq b$

We denote the **image** of g by γ

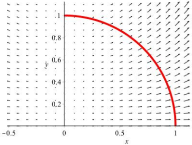
Definition The **Line Integral** of **F** over γ is

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = \int_a^b \mathbf{F}(g(t)) \cdot g'(t)\, dt$$

Example

Curve: $g(t) = (\cos t, \sin t), 0 \leq t \leq \frac{\pi}{2}$

Vector Field: $\mathbf{F}(x,y) = (x, yx^2)$



Then $\mathbf{F}(g(t)) = \mathbf{F}(\cos t, \sin t) = (\cos t, \sin t \cos^2 t)$ and

$g'(t) = (-\sin t, \cos t)$

Hence $\mathbf{F}(g(t)) \cdot g'(t) = (\cos t, \sin t \cos^2 t) \cdot (-\sin t, \cos t) =$

$-\sin t \cos t + \sin t \cos^2 t \cos t = -\sin t \cos t + \sin t \cos^3 t$

so $\int_\gamma \mathbf{F} = \int_0^{\pi/2} (-\sin t \cos t + \sin t \cos^3 t)\, dt$

$$= \left[\frac{\cos^2 t}{2} - \frac{\cos^4 t}{4} \right]_0^{\pi/2} = 0 - 0 - \frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$$

Alternative Notation for $n = 2$

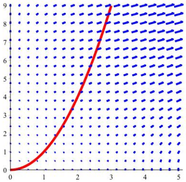
$$g(T) = (g_1(t), g_2(t)) = (x(t), y(t))$$

$$\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$$

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = \int_\gamma (F_1 dx + F_2 dy)$$

In our example, $\int_\gamma (x dx + yx^2 dy)$

Example: Find $\int_\gamma \mathbf{F}$ where $\mathbf{F}(x,y) = (2xy, x^2+2y)$ and γ is the graph of $y = x^2$ from $x = 0$ to $x = 3$.



Solution: First, find a parametrization of γ.
Here $g(t) = (t, t^2), 0 \le t \le 3$ will work.
Then $g'(t) = (1, 2t)$ and
$\mathbf{F}(g(t)) = F(t,t^2) = (2t^3, t^2+2t^2) = (2t^3, 3t^2)$
so $\mathbf{F}(g(t)) \cdot g'(t) = 2t^3 + 6t^3 = 8t^3$

and $\int_\gamma \mathbf{F} = \int_0^3 8t^3\,dt = 2t^4\Big|_0^3 = 162.$

What If We Used A Different Parametrization?

$$\mathbf{F}(x,y) = (2xy, x^2 + 2y)$$

Example: Let $h(t) = (\sqrt{t}, t)$ on $0 \leq t \leq 9$

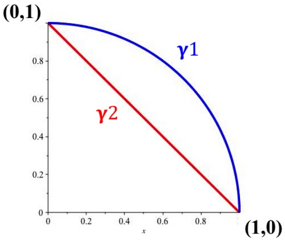
Then $h'(t) = (\frac{1}{2\sqrt{t}}, 1)$

Here $\mathbf{F}(h(t)) = \mathbf{F}((\sqrt{t}, t)) = (2t\sqrt{t}, t + 2t) = (2t^{3/2}, 3t)$

$$\int_\gamma \mathbf{F} = \int_0^9 [W] \; dt = \int_0^9 4t \, dt = 2t^2 \Big|_0^9 = 162$$

Theorem The value of the line integral $\int_\gamma \mathbf{F}$ is independent of the parametrization of γ but in general is dependent on the curve itself.

Proof: Use Change of Variable Formula; see text.



For some vector fields, the line integral $\int_\gamma \mathbf{F}$ depends only on the **endpoints** of the curve.

Theorem (**The Fundamental Theorem of Calculus for Line Integrals**). Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable and let $\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Then $\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$.

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Proof: Let g be any parametrization of γ. with $g(0) = \vec{a}$ and $g(1) = \vec{b}$.

Thus $\mathbb{R}^1 \to g \to \mathbb{R}^n \to f \to \mathbb{R}^1$

Use Our Old Friend **The Chain Rule**;

$$[f(g(t)]' = f'(g(t)) \cdot g'(t)$$
$$= \nabla f(g(t)) \cdot g'(t)$$
$$= \mathbf{F}(g(t)) \cdot g'(t)$$

Hence $\int_\gamma \mathbf{F} = \int_0^1 \mathbf{F}(g(t)) \cdot g'(t)\, dt$

$$= \int_0^1 [f(g(t)]'\, dt$$
$$= [f(g(t)]\Big|_0^1$$
$$= f(g(1)) - f(g(0)) = f(\vec{b}) - f(\vec{a})$$

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Then $\int_\gamma \mathbf{F} = \int_\gamma \mathbf{F} \nabla f = f(\vec{b}) - f(\vec{a})$.

Example: $f(x,y) = x^2 y + y^2$

so **F** $= \nabla f = (2xy, x^2 + y)$

let γ be any curve from (0,0) to (4,2)

Then $\int_\gamma \mathbf{F} = f(4,2) - f(0,0) = 4^2 \times 2 + 2^2 - (0+0) = 36$

If **F** $= \nabla f$ for some f, then we call **F** a **Conservative Vector Field**

and f is called a **Potential** of **F**

Many Applications of the Line Integral

Work

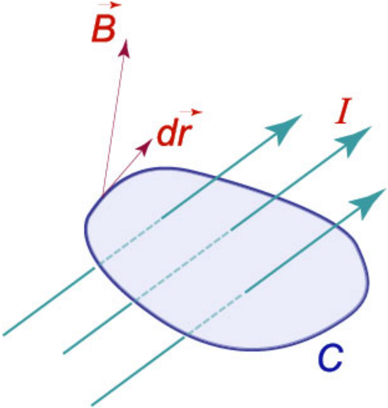
Position x along a line segment of a moving object is given by $x = g(t)$ where $g(0) =$ START and $g(T) =$ END.

$$\text{Work} = \int_{x=START}^{x=END} \mathbf{F}(x)\,dx = \int_0^T \mathbf{F}(g(t)) \cdot g'(t)\,dt$$

$x = g(t)$ implies $dx = g'(t)dt$

Other Physical Applications of Line Integrals

- Mass of a Wire
- Center of Mass and Moments of Inertia of a Wire;
- Magnetic Field Around a Conductor (**Ampere's Law)**: The line integral of a magnetic field **B** around a closed path C is equal to the total current flowing through the area bounded by the contour C

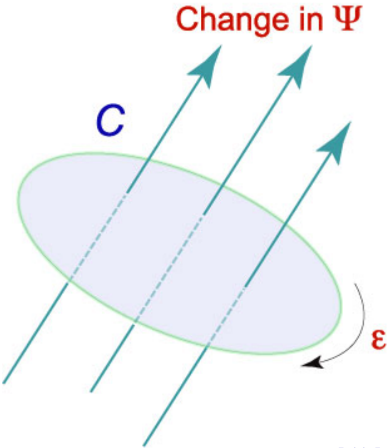


Other Physical Applications of Line Integrals

Voltage Generated in a Loop

(**Faraday's Law of Magnetic Induction**).

The electromotive force ϵ induced around a closed loop C is equal to the rate of the change of magnetic flux Ψ passing through the loop.



Applications in Economics

Buhr, Walter; Wagner, Josef

Working Paper

Line Integrals In Applied Welfare Economics: A Summary Of Basic Theorems

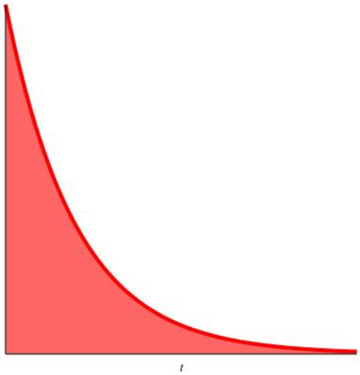
Volkswirtschaftliche Diskussionsbeiträge, No. 54-95

Link to Paper

An Important Example:
Exponential Probability Density Function

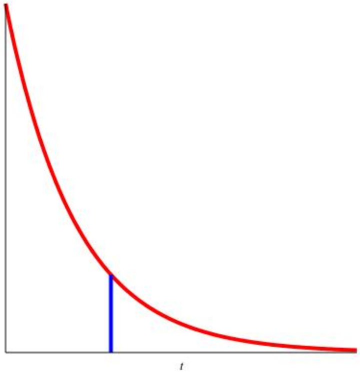
$$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_0^{b} e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\Big|_{x=0}^{b}\right]$$

$$= \lim_{b\to\infty}\left[-e^{-b} - (-e^{0})\right] = \lim_{b\to\infty}\left[1 - \frac{1}{e^{b}}\right] = 1$$



Exponential Probability Density Function

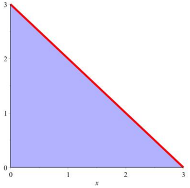
Probability(Light Bulb Burns Out in $\leq x$ months) $=$
$\int_0^x e^{-t}\, dt = 1 - e^{-x}$



x	$\int_0^x e^{-t} dt$	Prob(Bulb Lasts More than x months)
1	.632	.368
2	.865	.135
3	.950	.050
4	.982	.018

Suppose You Buy 2 Light Bulbs

What Is The Probability They Will Provide At Least 3 Months of Service?



Prob$(x + y > 3) = 1$ - Prob$(x + y \leq 3)$

$$= 1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$$

Evaluate $1-\int_{x=0}^{3}\int_{y=0}^{3-x} e^{-x}e^{-y}\,dy\,dx$

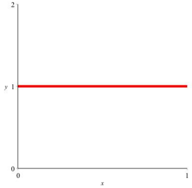
$$
\begin{aligned}
&= 1-\int_0^3 e^{-x}\left[-e^{-y}\Big|_{y=0}^{3-x}\right]\,dx\\
&= 1-\int_0^3 e^{-x}\left[-e^{3-x}+1\right]\,dx\\
&= 1-\int_0^3 (e^{-x}-e^{-3})\,dx\\
&= 1-\left[-e^{-x}-e^{-3}x\right]_{x=0}^{3}
\end{aligned}
$$

$$= 1-\left[-e^{-3}-3e^{-3}+1+0\right] = 1-\left[1-\frac{4}{e^3}\right] = \frac{4}{e^3} \approx .199$$

Probability Density Function

A real-valued function p such that $p(\vec{x}) \geq 0$ for all $\vec{x}$ and $\int_S p = 1$ where S is the set of all possibilities.

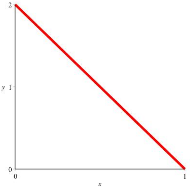
Example 1 Uniform Density: $p(x) = 1$ on [0,1]



$$\int_S p = \int_0^1 1 = x\Big|_0^1 = 1$$

Example 2: $p(x) = 2 - 2x$ on [0,1]

More likely to choose small numbers than larger numbers



Problem: Find the probability of picking a number less than 1/2.

$$\int_0^{1/2} (2-2x)\,dx = (2x - x^2)\Big|_0^{1/2} = (1 - \tfrac{1}{4}) - (0-0) = \tfrac{3}{4}$$

A probability density function on a set S in $\mathbb{R}^n$ is a continuous non-negative real-valued function $p : S \to \mathbb{R}^1$ such that

$$\int_S p dV = 1$$

If an experiment is performed where S is the set of all possible outcomes, then the probability that the outcome lies in a particular subset T is $\int_T p(\vec{x})\,dV$.

Example: Suppose two numbers b and c are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing b and c is equivalent to choosing a point (b, c) from the unit square S with $p(\vec{x}) = 1$ (**Uniform Density**)

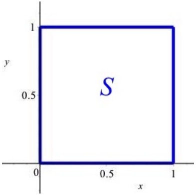
Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1$.

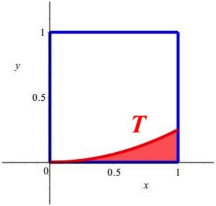
Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

$$\int_T p(\vec{x}) = \int_{x=0}^{1} \int_{y=0}^{x^2/4} 1 \, dy \, dx = \int_{x=0}^{1} \frac{x^2}{4} \, dx = \frac{x^3}{12}\Big|_0^1 = \frac{1}{12}$$





General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

$$\text{Prob(2 lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

$$\text{More than } b \text{ hours: } e^{-3b\lambda}(1 + b\lambda)$$