MATH 223: Multivariable Calculus

Notes on Class 2

September 13, 2023

Analog of Straight Line In Higher Dimensions

Line: $ax + by = c$

$a_1x_1 + a_2x_2 = c$

Plane: $a_1x_1 + a_2x_2 + a_3x_3 = d$ Other Important

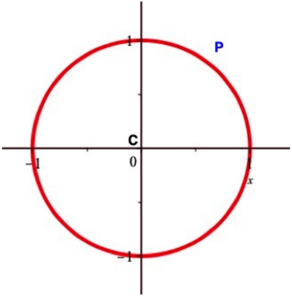
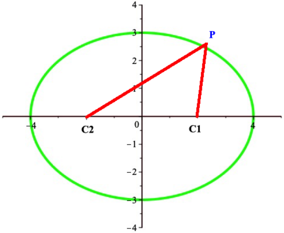
$ax + by + cz = d$

Hyperplane: $a_1x_1 + a_2x_2 + ... + a_nx_n = d$

Curves: CIRCLES and ELLIPSES
and the counterparts in higher dimensions.

Recall: Graph of $f : \mathcal{R}^1 \to \mathcal{R}^1$ is a curve in the plane.
BUT: NOT EVERY CURVE IN THE PLANE IS THE GRAPH OF SUCH A FUNCTION

Vertical Line Test

CIRCLE	ELLIPSE
	
Set of all points a fixed distance from a fixed point (center)	Set of all points, sum of distances to pair of fixed points is constant
distance(P, C) = r	$d(P, C_1) + d(P, C_2) = r$

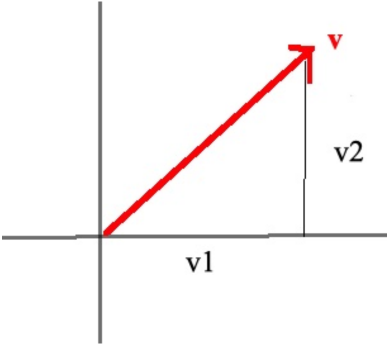
Distance in $\mathcal{R}^n$

Magnitude of a vector $\mathbf{v} = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Where does this come from?

Consider $\mathbf{v} = (v_1, v_2)$ in $\mathcal{R}^2$ where $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2$

$\sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2}$ (Pythagorean Theorem)



Distance Between x and y = | x - y |

Examine Circle in the Plane
Center $\mathbf{C} = (a, b)$ and radius r
Variable Point $\mathbf{P} = (x, y)$
Defining Relationship: $d(\mathbf{P}, \mathbf{C}) = r$
which means

$$\begin{aligned} |\mathbf{P} - \mathbf{C}| &= r \\ |(x - a, y - b)| &= r \\ \sqrt{(x-a)^2 + (y-b)^2} &= r \\ (x-a)^2 + (y-b)^2. &= r^2 \end{aligned}$$

Multiply Out: $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$
$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$
which has the form $x^2 + y^2 + Ax + By = C$.

Can We Go Backwards?

Example: $x^2 + y^2 - 6x + 16y = 71$

Complete the Squares in x and y

$$(x^2 - 6x) + (y^2 + 16y) = 71$$

$$(x^2 - 6x + 9) + (y^2 - +16y + 64) = 71 + 9 + 64$$

$$(x-3)^2 + (y+8)^2 = 144 = 12^2$$

Circle as Image of a ffunction $f : \mathcal{R}^1 \to \mathcal{R}^2$

Parametrization with parameter t

Example: $\mathbf{f}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$

Example:

$$\begin{Bmatrix} x = 12\cos t + 3 \\ y = 12\sin t - 8 \end{Bmatrix}$$

$$\begin{Bmatrix} x - 3 = 12\cos t \\ y + 8 = 12\sin t - 8 \end{Bmatrix}$$

$$(x-3)^2 + (y+8)^2 = 12^2$$

$$\mathbf{f}(t) = (12\cos t + 3, 12\sin t - 8)$$

ELLIPSE

Standard Ellipse:

Center at (0,0)

Foci at $(\pm c, 0)$

Vertices $(\pm a, 0)$ and $(0, \pm b)$

(a,0) distance from (c,0) + distance from (-c,0)

(a - c) + (a - (-c) = a -c + a + c = 2a

(0,b) distance from (c,0) + distance from (-c,0)

$\sqrt{c^2 + b^2} + \sqrt{c^2 + b^2} = 2\sqrt{c^2 + b^2}$

Thus $2\sqrt{c^2 + b^2} = 2a$

So $c^2 + b^2 = a^2$ implying $c^2 = a^2 - b^2$

(x,y) distance from (c,0) + distance from (-c,0) = 2a

$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a$

Much algebra yields

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametrization: $\mathbf{g}(t) = (a \cos t, b \sin t), 0 \leq t \leq 2\pi$

The Algebra

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$$

Write as $\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$

Square Both Sides

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Expand, Simplify, and Divide by 4

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$$

$$cx-a^2=-a\sqrt{(x-c)^2+y^2}$$

Begin with $cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$

Square Again

$$c^2x^2 - 2a^2cx + a^4 = a^2\left((x-c)^2 + y^2\right)$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$
$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

Write as $(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2 = a^2(a^2 - c^2)$

But $a^2 - c^2 = b^2$ so

$$b^2x^2 + a^2y^2 = a^2b^2$$

Divide by a^2b^2:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$