MATH 223: Multivariable Calculus

Class 18: October 23, 2023

Handouts

Notes on Assignment 16

Assignment 17

Extreme Values

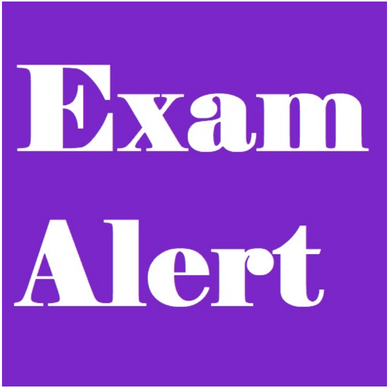
Announcements

Office Hours Today: 10:40 – 12:10

Chris Desjardins, Saint Michael's College

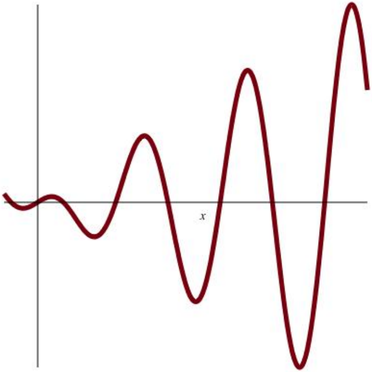
TODAY: 12:30–1:20 PM · Davis Family Library 105 **"Getting Tidy on Monday: Let's do Data Science"**

TidyTuesday is a weekly social data science project. Each week a new topic is posted and participants explore a data set, create visualizations, fit models, and share their findings on social media. Using a TidyTuesday dataset, in this talk, we will collaboratively complete a data science project. I will begin by briefly presenting my research interests and experiences and then we will embark on a data science project where we will identify a data set, perform exploratory data analysis using Shiny, create a research question, fit and evaluate a regression model in R, and provide an answer to our question. As time permits, we will share these findings on GitHub.



Wednesday, November 1

Today:
Maxima and Minima of Real-Valued Functions

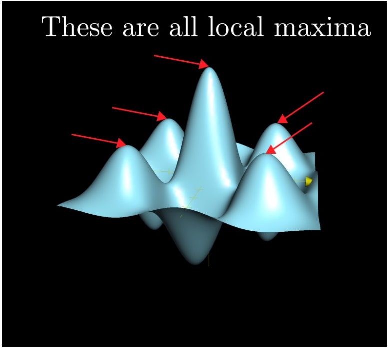


Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^1$ be a real-valued function with $\vec{x_o}$ a point in D.

<u>Definition</u>: f has an **absolute maximum** at $\vec{x_o}$ if $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in D.

Note: $\geq$ makes sense because we are comparing real numbers.

f has a **relative maximum** at $\vec{x_o}$ if there is a neighborhood N around $\vec{x_o}$ such that $f(\vec{x_o}) \geq f(\vec{x})$ for all $\vec{x}$ in N.



Theorem: Let $\vec{x_o}$ be an **interior** point of D. If f is differentiable at $\vec{x_o}$ and f has a relative maximum or minimum at $\vec{x_o}$, then $f'(\vec{x_o}) = \nabla(\vec{x_o}) = \vec{0}$.

Proof: Suppose f has a relative maximum at $\vec{x_o}$

Let $\vec{u}$ be any unit vector in $\mathbb{R}^n$.

$$\text{Then } \frac{\partial f}{\partial \vec{u}} = \lim_{t \to 0} \frac{f(\vec{x_0} + t\vec{u}) - f(\vec{x_0})}{t}$$

(a) Take $\lim_{t \to 0^+} : \frac{-}{+} \leq 0$

thus $\frac{\partial f}{\partial \vec{u}} = 0$ for all $\vec{u}$

(b) Take $\lim_{t \to 0^-} : \frac{-}{-} \geq 0$

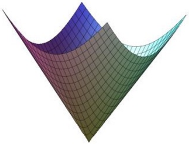
Taking $\vec{u}$ to be unit vectors gives $\nabla f(\vec{x_0}) = 0$

Theorem: f differentiable at relative extrema implies gradient is 0.

The Theorem Has Its Limitations:

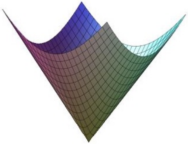
(1) **The function can have an extreme value at a point where it is not differentiable.**

Example: $f(x,y) = \sqrt{x^2+y^2}$ has minimum at (0,0) but is not differentiable there.



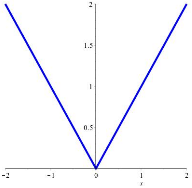
$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}, f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$

Example: $f(x, y) = \sqrt{x^2 + y^2}$ has minimum at (0,0) but is not differentiable there.



Analogue in Calculus I:

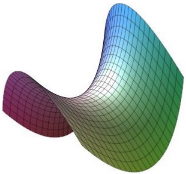
$f(x) = \sqrt{x^2} = |x|$

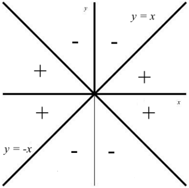


Theorem: f differentiable at relative extrema implies gradient is 0.

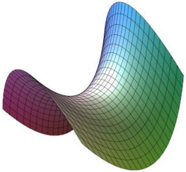
The Theorem Has Its Limitations:

(2) We can have $\nabla f(\vec{x_0}) = 0$ but no extreme point at $\vec{x_0}$





$$\nabla f(x, y) = (2x, 2y)$$



There is a Maximum is one direction and a Minimum in another

Saddle Point

Quiz:
Name a Famous Commercial Food Product That Exhibits a Saddle Point

Definition: A point $\vec{x_0}$ in the domain of f is a **Critical Point** of f if

(a) $\nabla f(\vec{x_0}) = \vec{0}$

or

(b) ∇f does not exist at $\vec{x_0}$.

Extreme Values Can Occur at Critical Points or Points on the Boundary

Example: Temperature Distribution on disk of radius 1 centered at origin is $T(x,y) = 2x^2 + 4y^2 + 2x + 1$.

For Critical Points, examine $\nabla T = (4x+2, 8y)$

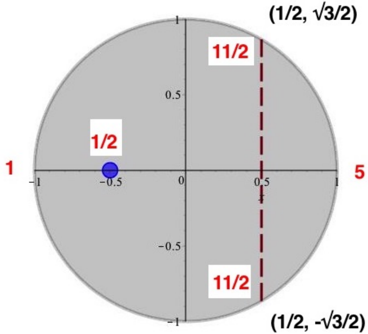
$\nabla T = (0,0)$ only at $x = -\frac{1}{2}, y = 0$

which does lie inside the disk.

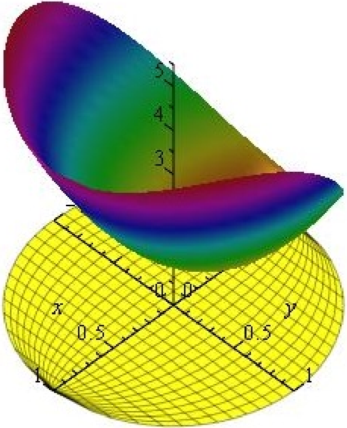
Note $T(-\frac{1}{2}, 0) = 2(\frac{1}{4}) + 4(0^2) + 2(-\frac{1}{2}) + 1 = \frac{1}{2}$, and $T(0,0) = 1$.

Analyze Along Boundary:

$$x^2 + y^2 = 1 \text{ so } y^2 = 1 - x^2 \text{ and}$$
$$T(x,y) = g(x) = 2x^2 + 4(1 - x^2) + 2x + 1 = -2x^2 + 2x + 5$$
Thus $g'(x) = -4x + 2, g''(x) = -4$ so $x = \frac{1}{2}$ gives a maximum.
$$x = \frac{1}{2} \text{ gives } y^2 = 1 - \frac{1}{4} = \frac{3}{4} \text{ so } y = \pm\frac{\sqrt{3}}{2}$$



red numbers are values of the function



5
4
3
0
0
x
y
0.5
0.5
1
1

Parametrize Boundary

$x = \cos t, y = \sin t$ for $0 \leq t \leq 2\pi$

$$
\begin{aligned}
T(x, y) &= 2x^2 + 4y^2 + 2x + 1 \\
&= 2\cos^2 t + 4\sin^2 t + 2\cos t + 1 \\
&= 2\cos^2 t + 2\sin^2 t + 2\sin^2 t + 2\cos t + 1 \\
&= 2 + 2\sin^2 t + 2\cos t + 1 = 2\sin^2 t + 2\cos t + 3 \\
&= H(t)
\end{aligned}
$$

$H(0) = 2 \cdot 1 + 2 \cdot 0 + 3 = 5, H(\pi) = 2 \cdot 1 + 2 \cdot -1 + 3 = 1$

Now $H'(t) = 4\sin t\cos t - 2\sin t = 2\sin t(2\cos t - 1)$ so

$H'(t) = 0$ at $\sin t = 0$ or $\cos t = \frac{1}{2}$

The first condition gives $t = 0, t = \pi$, the second occurs when $t = \frac{\pi}{3}$.

Next Time:

Solving Constrained Optimization Problems Using Lagrange Multipliers