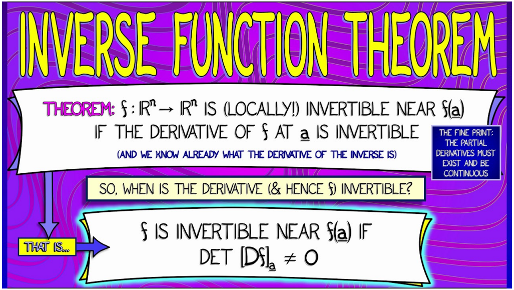
MATH 223: Multivariable Calculus



Class 15: October 16, 2023

- Notes on Assignment 13
- Assignment 14

The Mathematics of Opinion Dynamics
Heather Zinn-Brooks, Harvey Mudd College
Today, October 16 from 3:45–4:45 in Warner 101

Given the large audience and the ease of sharing content, the shifts in opinion driven by online interaction have important implications for interpersonal interactions, public opinion, voting, and policy. Mathematical models can help develop a theory to understand the mechanisms underpinning the spread of content and diffusion of information. In this talk, I'll introduce you to a variety of mathematical models for opinion dynamics.

Review Chain Rule

Implicit Differentiation II

Change of Variable

Inverse Function Theorem

Gradient Fields

The Chain Rule

$$\begin{tikzcd}
 & \mathcal{R}^m \arrow[dr, bend left, "g"] & \\
\mathcal{R}^n \arrow[ur, bend left, "f"] \arrow[rr, bend right, "g \circ f"'] & & \mathcal{R}^p
\end{tikzcd}$$

$$(g \circ f)' = g'(f(x))f'(x)$$

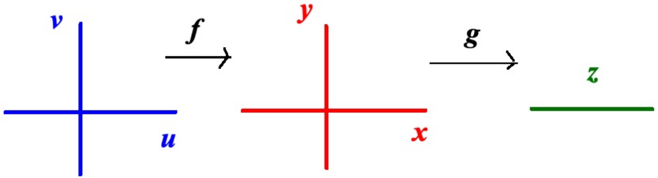
(p x m) (m x n)

matrix matrix

p x n matrix

Another Example: Suppose $x = u^2 - v^2, y = 2uv$ and $z = g(x, y)$ for some real-valued differentiable function g.

Show $(z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$



Another Example: Suppose $x = u^2 - v^2, y = 2uv$ and $z = g(x, y)$ for some real-valued differentiable function g.

$$\text{Show } (z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$$

$$\text{Let } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u^2 - v^2 \\ 2uv \end{pmatrix} = f\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\text{Then } f'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}, g'\begin{pmatrix} x \\ y \end{pmatrix} = (g_x, g_y) = (z_x, z_y)$$

$$\text{Now } (g \circ f)' = g'(f)f' = (z_x, z_y)\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} =$$

$$(2uz_x + 2vz_y, -2vz_x + 2uz_y) = (z_u, z_v)$$

Thus

$$\begin{aligned} z_u^2 + z_v^2 &= 4u^2z_x^2 + 8uvz_xx_y + 4v^2z_y^2 + 4v^2z_x^2 - 8uvz_xz_y + 4u^2u_z^2 \\ &= 4u^2(z_x^2 + z_y^2) + 4v^2(z_x^2 + z_y^2) = 4(u^2 + v^2)(z_x^2 + z_y^2) \end{aligned}$$

Implicit Differentiation

Example: Find slope of tangent line to the graph of
$4x^2 + 5y^2 = 61$ at (2,3).
(Check point lies on curve: $4(2^2) + 5(3^2) = 16 + 45 = 61$)

A: Direct Solution

$$5y^2 = 61 - 4x^2 \Rightarrow y^2 = \frac{61 - 4x^2}{5} \Rightarrow y = \sqrt{\frac{61 - 4x^2}{5}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{61 - 4x^2}{5}\right)^{-1/2} \frac{-8x}{5}$$

Evaluate at $x = 2$: to get $\frac{1}{2}\left(\frac{45}{5}\right)^{-1/2} \frac{-16}{5} = -\frac{8}{15}$

Implicit Differentiation

Example: Find slope of tangent line to the graph of $4x^2 + 5y^2 = 61$ at (2,3).

B: Classic Implicit Differentiation

Treat y as an unknown function of x and differentiate:

$$8x + 10y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-8x}{10y} = -\frac{4x}{5y}$$

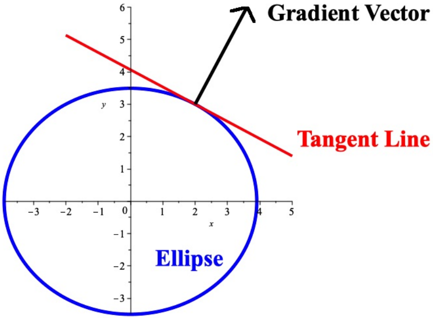
Evaluate at $x = 2, y = 3$: to get $-\frac{8}{15}$

C: Use Level Curve Idea

If $f(x,y) = 4x^2 + 5y^2$, then (2,3) lies on level curve $f(x,y) = 61$. Then $\nabla f(2,3)$ is normal to the curve so slope of tangent line is the negative of the slope of the gradient.

$\nabla f(x,y) = (8x, 10y)$ has slope $\frac{10y}{8x} = \frac{15}{8}$ at (2,3). Hence slope of tangent line is $-\frac{8}{15}$.

Example: Find slope of tangent line to the graph of
$4x^2 + 5y^2 = 61$ at (2,3).



The ellipse is the level curve $F(x, y) = 61$ or $F(x, y) - 61 =$ where
$F(x, y) = 4x^2 + 5y^2$.

A piece of the curve around (2,3) is the graph of some implicit function $y = f(x)$.

We want $f'(2)$.

Define a new function $\mathbf{G} : \mathcal{R}^1 \to \mathcal{R}^2$ by

$$\mathbf{G}(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix} \text{ so } \mathbf{G}'(x) = \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$$

Note that this is the tangent vector.

Then $(F \circ \mathbf{G})(x) = 61$ for all x

Take Derivative Using The Chain Rule:

$$F'(\mathbf{G}(x))\mathbf{G}'(x) = 0. \text{Thus } \nabla F(\mathbf{G}(x) \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} = 0$$

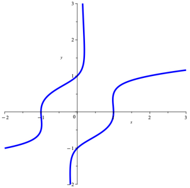
Now $G(2) = 3$ and $F(x, y) = 4x^2 + 5y^2$ implies
$\nabla F(x, y) = (8x, 10y)$.
Hence $\nabla F(G(2)) = (8 \times 2, 10 \times 3) = (16, 30)$.
We have $(16, 30) \begin{pmatrix} 1 \\ f'(2) \end{pmatrix} = 0$ so $16 + 30f'(2) = 0$ and thus
$f'(2) = -16/30 = -8/15$.

Gradient Vector Is Orthogonal To Tangent Line at Level Curve

Normal

Another Example of Implicit Differentiation

Find Slope of Tangent Line to Graph of $x^2 + y^2 - 2xy^3 = 1$ at (2,1)



Method I: Solve $x^2 + y^2 - 2 * x * y^3 = 1$ for y in terms of x?

Method II: Differentiate In Place?

Method III: Use Gradient Vector:

$$\nabla F(x, y) = (2x - 2y^3, 2y - 6xy^2) \text{ so} \nabla F(2, 1) = (2, -10)$$

Slope of Gradient = =10/2 = -5 so Slope of Tangent Line is 1/5

Change of Variable

Example: Find $\int (10x+15)^{1/3} dx$

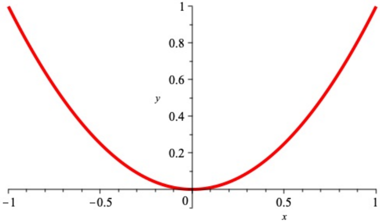
Change of Variable $u = 10x + 15$ so $\mathbf{x} = \frac{\mathbf{u-15}}{\mathbf{10}}$ and $dx = \frac{1}{10} du$

Integral becomes $$\int (10x+15)^{1/3} dx = \int u^{1/3} \frac{1}{10} du = \frac{1}{10} \int u^{1/3} du$$
$$= \frac{1}{10} \times \frac{3}{4} u^{4/3} + C$$
$$= \frac{3}{40} (10x+15)^{4/3} + C$$

$\mathbf{x} = \frac{\mathbf{u-15}}{\mathbf{10}}$ is key step. WE MUST BE ABLE TO INVERT THE SUBSTITUTION.

Change of Variable should be invertible, a one-to-one function.

Not Every Function is Invertible

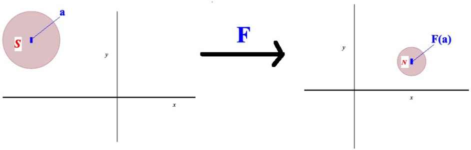


If $y = x^2$, we can not solve unambiguously for x in terms of y globally

$$x = \pm\sqrt{y}$$

but we can solve locally except at origin.

Inverse Function Theorem for $\mathbf{f}: \mathcal{R}^n \to \mathcal{R}^n$



IF

- $\mathbf{a}$ is a point in $\mathcal{R}^n$
- S is an open set containing $\mathbf{a}$
- $\mathbf{f}$ is continuously differentiable on S
- Derivative Matrix $\mathbf{f}'(\mathbf{a})$ is invertible

Then

There is a neighborhood N of $\mathbf{a}$ on which $\mathbf{f}^{-1}$ is defined and

$$\left(\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}))\right)' = [\mathbf{f}'(\mathbf{x})]^{-1} \text{ for all } \mathbf{x} \text{ in } N$$

Example: $\mathbf{f}(x,y) = (\cos x, x\cos x - y)$

$$J = \mathbf{f}'(x,y) = \begin{pmatrix} -\sin x & 0 \\ \cos x - x\sin x & -1 \end{pmatrix}$$

$\det J = \sin x$ so we have invertibility if $x \neq 0, \pi$.

$$(\mathbf{f}^{-1}(x,y))' = J^{-1} = \begin{pmatrix} \frac{-1}{\sin x} & 0 \\ \frac{x\sin x - \cos x}{\sin x} & -1 \end{pmatrix}$$

At $x = \pi/6, y = 2$:

$$f\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\frac{\sqrt{3}}{2} - 2\right) = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}\,\pi}{12} - 2\right)$$

and

$$\mathbf{f}^{-1}(\pi/6, 2))' = \begin{pmatrix} -2 & 0 \\ \frac{\pi}{6} - \sqrt{3} & -1 \end{pmatrix}$$

Gradient Fields

A Gradient Field is just a function from $\mathcal{R}^n$ to $\mathcal{R}^n$ which is the gradient of a differentiable real-valued function.

The gradient $\nabla f(x, y)$ of $f : \mathcal{R}^2 \to \mathcal{R}^2$.

Example 1: $f(x, y) = x^2 \sin y$

Here $\nabla f(x, y) = (2x, x^2 \cos y) = (f_x(x, y), f_y(x, y))$

Note $f_{xy} = x^2 \cos y = f_{yx}$ [Equality of Mixed Partials]

Example 2: Is $\mathbf{F}(x, y) = (y, 2x)$ a gradient field?

If $\mathbf{F} = \nabla f$, then

$$f_x(x, y) = y \implies f_{xy}(x, y) = 1$$
$$f_y(x, y) = 2x \implies f_{yx}(x, y) = 2$$

But these are not equal!

What if we try to build an f by "Partial Integration"?

$$f_x(x, y) = y \implies f(x, y) = xy + G(y) \implies f_y(x, y) = x + G'(y)$$

but we would need G a function of y such that $G'(y) = x$.

We can work backwards on Example 1:
Given $f_x(x, y) = 2x \sin y$, "partial integration" with respect to x produces $f(x, y) = x^2 \sin y + G(y)$ and that yields $f_y = x^2 \cos y + G'(y)$ which equals $x^2 \cos y$ by choosing G to be any constant function.