MATH 223: Multivariable Calculus

Class 14: October 11, 2022

▶ Notes on Assignment 12

▶ Assignment 13

Today

Generalized Mean Value Theorem

Chain Rule

Implicit Differentiation

Mean Value Theorem for $f : \mathcal{R}^n \to \mathcal{R}^1$

If f is differentiable at each point of a line segment S between **a** and **b**, then there is a least point **c** on S such that
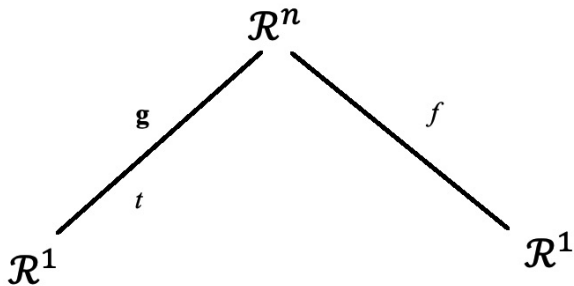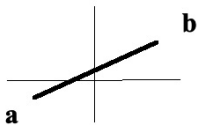
$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Recall classic MVT from Single Variable Calculus:

If $f : \mathcal{R}^1 \to \mathcal{R}^1$ is differentiable on a closed interval $[a, b]$, then there is at least on c inside the interval such that

$$f(b) - f(a) = f'(c)(b - a).$$

Generalized Mean Value Theorem

Proof of Generalized Mean Value Theorem

Define a new function $\mathbf{g} : [0, 1] \to \mathcal{R}^n$ by $\mathbf{g}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

Note $\mathbf{g}(0) = \mathbf{a}$ and $\mathbf{g}(1) = \mathbf{b}$ and $\mathbf{g}(t)$ lies on S and $\mathbf{g}'(t) = \mathbf{b} - \mathbf{a}$

Consider the composition $H(t) = f(\mathbf{g}(t)) : [0, 1] \to \mathcal{R}^1$

Apply Classic MVT to H:

$$H(1) - H(0) = H'(t_c)(1 - 0) = H'(t_c)$$

but $H(1) = f(g(1)) = f(\mathbf{b})$ and $H(0) = f(g(0)) = f(\mathbf{a})$

Thus $f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c)$

What is $H'(t)$? By Chain Rule: $f'(\mathbf{g}(t))\mathbf{g}'(t) = \nabla f(\mathbf{g}(t)) \cdot (\mathbf{b} - \mathbf{a})$
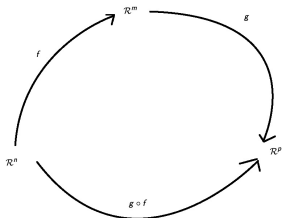
Let $\mathbf{C} = \mathbf{g}(t_c)$. Then

$$f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c) = \nabla f(\mathbf{C}) \cdot (\mathbf{b} - \mathbf{a})$$

An Important Consequence of classic MVT:

Suppose $f'(x) = g'(x)$ for all x in $[a, b]$. Then $f(x) = g(x) + C$

for some constant C and all x in the interval.

Proof: Last Time

The Chain Rule

$$(g \circ f)' = g'(f(x))f'(x)$$

(p x m) (m x n)

matrix matrix

p x n matrix

Example Find $(g \circ f)'$ at $(2,3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ if

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix}, g \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ 2u \\ v^2 \end{pmatrix}$$

Step I: $f \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2^2 + 6 + 1 \\ 9 + 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$

Step II: $(g \circ f)' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = g' \left(f \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) f' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = g' \begin{pmatrix} 11 \\ 11 \end{pmatrix} f' \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$g' \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2v \end{pmatrix} \text{ so } g' \begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}$$

$$f' \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x \\ 0 & 2y \end{pmatrix} \text{ so } f' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}$$

$$(g \circ f)' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 14 & 4 \\ 0 & 132 \end{pmatrix}$$

$$(g \circ f)' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 14 & 4 \\ 0 & 132 \end{pmatrix}$$

Here we can actually check by direct computation:

$$g(f(x,y)) = g \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 + y^2 + 2 \\ 2x^2 + 2xy + 2 \\ y^4 + 4y^2 + 4 \end{pmatrix}$$

$$(g \circ f)' \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x + 2y \\ 4x + 2y & 2x \\ 0 & 4y^3 + 8y \end{pmatrix}$$

$$(g \circ f)' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 + 3 & 2 + 6 \\ 8 + 6 & 4 \\ 0 & 108 + 24 \end{pmatrix} = \begin{pmatrix} 7 & 2 + 6 \\ 14 & 4 \\ 0 & 132 \end{pmatrix}$$

<u>Another Example</u>: Suppose $x = u^2 - v^2, y = 2uv$ and $z = g(x, y)$
for some real-valued differentiable function g.

Show $(z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$
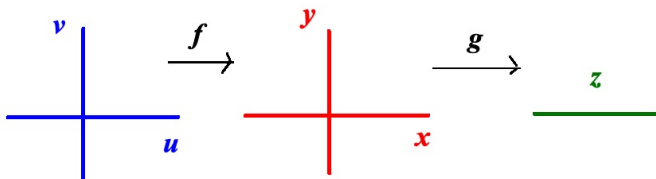
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$$\text{Let } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u^2 - y^2 \\ 2uv \end{pmatrix} = f \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\text{Then } f' \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}, g' \begin{pmatrix} x \\ y \end{pmatrix} = (g_x, g_y) = (z_x, z_y)$$
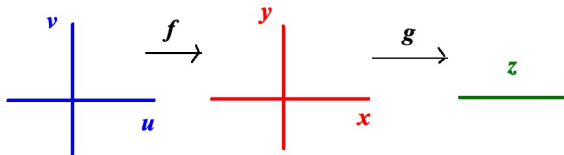
$$\text{Now } (g \circ f)' = g'(f)f' = (z_x, z_y) \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} =$$

$$(2uz_x + 2vz_y, -2vz_x + 2uz_y) = (z_u, z_v)$$

Thus

$$z_u^2 + z_v^2 = 4u^2 z_x^2 + 8uv z_x x_y + 4v^2 z_y^2 + 4v^2 z_x^2 - 8uv z_x z_y + 4u^2 u_z^2$$
$$= 4u^2(z_x^2 + z_y^2) + 4v^2(z_x^2 + z_y^2) = 4(u^2 + v^2)(z_x^2 + z_y^2)$$

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Implicit Differentiation

Example: Find slope of tangent line to the graph of
$$4x^2 + 5y^2 = 61 \text{ at } (2,3).$$
(Check point lies on curve: $4(2^2) + 5(3^2) = 16 + 45 = 61$)

A: Direct Solution

$$5y^2 = 61 - 4x^2 \Rightarrow y^2 = \frac{61 - 4x^2}{5} \Rightarrow y = \sqrt{\frac{61 - 4x^2}{5}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{61 - 4x^2}{5}\right)^{-1/2}\frac{-8x}{5}$$

Evaluate at $x = 2$: to get $\frac{1}{2}\left(\frac{45}{5}\right)^{-1/2}\frac{-16}{5} = -\frac{8}{15}$

Implicit Differentiation

<u>Example</u>: Find slope of tangent line to the graph of
$$4x^2 + 5y^2 = 61 \text{ at } (2,3).$$

B: Classic Implicit Differentiation

Treat y as an unknown function of x and differentiate:

$$8x + 10y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-8x}{10y} = -\frac{4x}{5y}$$

Evaluate at $x = 2, y = 3$: to get $-\frac{8}{15}$

C: Use Level Curve Idea

If $f(x, y) = 4x^2 + 5y^2$, then (2,3) lies on level curve $f(x, y) = 61$.
Then $\nabla f(2, 3)$ is normal to the curve so slope of tangent line is the negative of the slope of the gradient.
$\nabla f(x, y) = (8x, 10y)$ has slope $\frac{10y}{8x} = \frac{15}{8}$ at (2,3). Hence slope of tangent line is $-\frac{8}{15}$.