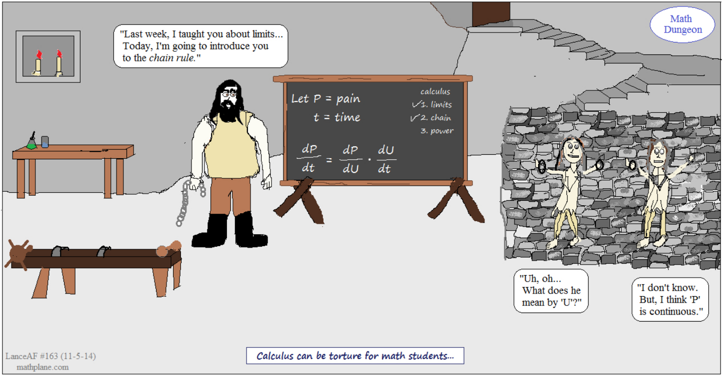
MATH 223: Multivariable Calculus



Class 13: October 9, 2023

- Notes on Assignment 11
- Assignment 12

MATHEMATICS TALK

Pete Schumer

"Math Puzzles with Hats and Lockers"

3:45 PM Tuesday

Warner 101

Refreshments

Partial With Respect to a Vector

$$f : \mathcal{R}^n \to \mathcal{R}^1$$

a point and **v** vector in $\mathcal{R}^n$

The partial derivative $f_{\mathbf{v}}(\mathbf{a})$ of f at **a** if we approach **a** along vector **v**

$$\text{We want } f_{\mathbf{v}}(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}$$

Theorem: If f is differentiable at **a**, then

$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Theorem: If $f : \mathcal{R}^n \to \mathcal{R}^1$ is differentiable at $\mathbf{a}$, then
$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Proof of Theorem:
(Case 1): $\mathbf{v} = \mathbf{0}$: Both sides are 0.
(Case 2): $\mathbf{v} \neq \mathbf{0}$:
Note: $|\mathbf{v}| \neq 0$ so we can divide by $|\mathbf{v}|$ if necessary.
By differentiability of f at $\mathbf{a}$, we have

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = 0$$

Set $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ so $\mathbf{x} \to \mathbf{a}$ is equivalent to $t \to 0$ and $\mathbf{x} - \mathbf{a} = t\mathbf{v}$
We have
$$\lim_{t\to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot t\mathbf{v}}{|t\mathbf{v}|} = 0$$

$$\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a}) - \nabla f(\mathbf{a})\cdot t\mathbf{v}}{|t\mathbf{v}|} = 0$$

Now $|t\mathbf{v}| = |t||\mathbf{v}|$
Can take $t > 0$ (Why?). So $|t\mathbf{v}| = t|\mathbf{v}|$
We can write limit as

$$\lim_{t\to 0} \left[\frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t|\mathbf{v}|} - \frac{t\nabla f(\mathbf{a})\cdot \mathbf{v}}{t|\mathbf{v}|}\right] = 0$$

Factor out t from second term and multiply both sides by the nonzero scalar $|\mathbf{v}|$ t to obtain

$$\lim_{t\to 0} \left[\frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t} - \nabla f(\mathbf{a})\cdot \mathbf{v}\right] = 0$$

$$\lim_{t\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a})}{t}-\nabla f(\mathbf{a})\cdot\mathbf{v}\right]=0$$

implies

$$\lim_{t\to 0}\left[\frac{f(\mathbf{a}+t\mathbf{v})-f(\mathbf{a})}{t}\right]=\nabla f(\mathbf{a})\cdot\mathbf{v}$$

But the left hand side is, by definition $f_{\mathbf{v}}(\mathbf{a})$

Directional Derivative

$$f : \mathcal{R}^n \to \mathcal{R}^1$$

a point and **v** vector in $\mathcal{R}^n$

Find the directional derivative of f at**a** in the direction of the vector **v** is

$$f_{\mathbf{u}}(\mathbf{a}) \text{ where } \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Rate of Change in Direction **u** is

$$\nabla f(\mathbf{a}) \cdot \mathbf{u} = |\nabla f(\mathbf{a})||\mathbf{u}| \cos\theta = |\nabla f(\mathbf{a})| \cos\theta$$

since $|\mathbf{u}| = 1$.

Maximum rate of change occurs when $\cos\theta = 1$; that is $\theta = 0$ so pick **u** in the direction of the gradient.

Mean Value Theorem for $f : \mathcal{R}^n \to \mathcal{R}^1$

If f is differentiable at each point of a line segment S between **a** and **b**, then there is a least point **c** on S such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Recall classic MVT from Single Variable Calculus:

If $f : \mathcal{R}^1 \to \mathcal{R}^1$ is differentiable on a closed interval $[a, b]$, then there is at least on c inside the interval such that

$$f(b) - f(a) = f'(c)(b - a).$$

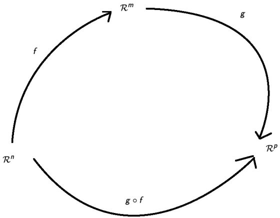
An Important Consequence of classic MVT:
Suppose $f'(x) = g'(x)$ for all x in $[a, b]$. Then $f(x) = g(x) + C$ for some constant C and all x in the interval.

Proof: Let $H(x) = f(x) - g(x)$.
Then $H'(x) = f'(x) - g'(x) = 0$ for all x in the interval.
Now let $x_1 < x_2$ be any two points in the interval.
By MVT: $H(_2) - H(x_1) = H'(c)(x_2 - x_1) = 0(x_2 - x_1) = 0$. Thus H is a constant function: $H(x) = C$ for all x.
So $f(x) - g(x) = C$ and hence $f(x) = g(x) + C$.

The same argument shows

$$\nabla f \equiv \nabla g \text{ implies } f(\mathbf{x}) = g(\mathbf{x}) + C$$

The Chain Rule



$$(g \circ f)' = g'(f(x))f'(x)$$

(p x m) (m x n)

matrix matrix

p x n matrix

Example Find $(g \circ f)'$ at $(2,3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ if

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix}, g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ 2u \\ v^2 \end{pmatrix}$$

Step I: $f\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2^2 + 6 + 1 \\ 9 + 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$

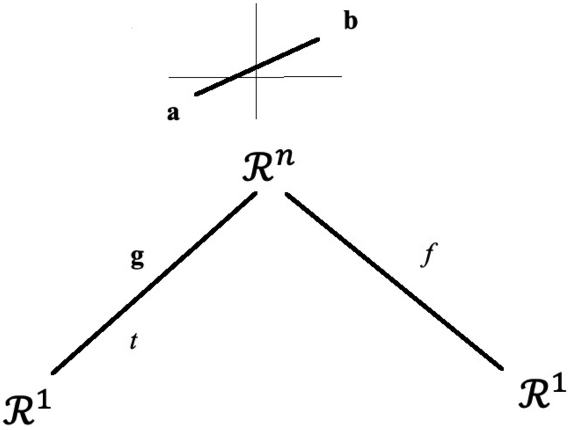
Step II: $(g \circ f)'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = g'\left(f\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} f'\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$g'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2v \end{pmatrix}$ so $g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}$

$f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x \\ 0 & 2y \end{pmatrix}$ so $f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}$

$$(g \circ f)'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 14 & 4 \\ 0 & 132 \end{pmatrix}$$

Generalized Mean Value Theorem



Proof of Generalized Mean Value Theorem

Define a new function $\mathbf{g} : [0,1] \rightarrow \mathcal{R}^n$ by $\mathbf{g}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$
Note $\mathbf{g}(0) = \mathbf{a}$ and $\mathbf{g}(1) = \mathbf{b}$ and $\mathbf{g}(t)$ lies on S and $\mathbf{g}'(t) = \mathbf{b} - \mathbf{a}$

Consider the composition $H(t) = f(\mathbf{g}(t)) : [0,1] \rightarrow \mathcal{R}^1$
Apply Classic MVT to H:

$$H(1) - H(0) = H'(t_c)(1-0) = H'(t_c)$$

but $H(1) = f(g(1)) = f(\mathbf{b})$ and $H(0) = f(g(0)) = f(\mathbf{a})$
Thus $f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c)$
What is $H'(t)$? By Chain Rule: $f'(\mathbf{g}(t))\mathbf{g}'(t) = \nabla f(\mathbf{g}(t)) \cdot (\mathbf{b} - \mathbf{a})$
Let $\mathbf{C} = \mathbf{g}(t_c)$. Then

$$f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c) = \nabla f(\mathbf{C}) \cdot (\mathbf{b} - \mathbf{a})$$