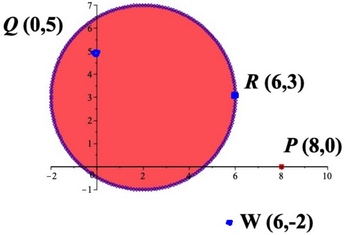
MATH 223: Multivariable Calculus



Class 11: October 4, 2023

Announcements

Exam 1: TONIGHT, 7 PM -
No Time Limit

Warner 101

No Books, Computers, Smartphones, etc.
One Page of Notes OK
Focus on Chapters 2 and 3

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(x) = \mathbf{b}$$

means

For every ϵ-neighborhood V of $\mathbf{b}$, there is an δ-neighborhood U of $\mathbf{a}$ such that $\mathbf{x}$ in U ($\mathbf{x} \neq \mathbf{a}$) implies $\mathbf{f}(\mathbf{x})$ is in V.

A function $\mathbf{f}$ is **continuous** at $\mathbf{a}$ if there is a $\mathbf{b}$, such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$$

and

$$\mathbf{f}(\mathbf{a}) = \mathbf{b}$$

Today: Begin Chapter 4

Topic: Differentiability

Start with $f : \mathcal{R}^n \to \mathcal{R}^1$

Eventually: $\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$

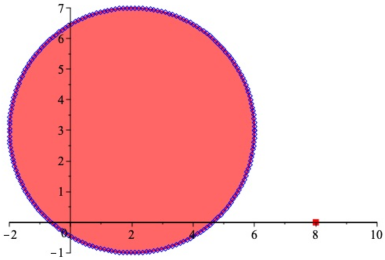
Derivative at point turns out to be $m \times n$ matrix.

But First: Limits and Continuity

Limits and Continuity: Preliminary Concepts

Open Set	Interior Point
Closed Set	Boundary Point
Limit Point	Neighborhood

Example: $S = \{|x-(2,3)| < 4\} \cup \{(8,0)\}$



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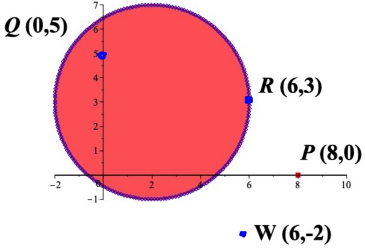

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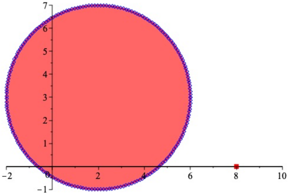

Point	Interior Point?	Limit Point	Boundary Point
Q	Yes	Yes	Yes
R	No	Yes	Yes
P	No	No	Yes
W	No	No	No

Differentiability = Local Linearity = Approximatable By Tangent Object

$$
\begin{aligned}
f(x) &\approx f(a) + f'(a)(x-a) \\
\text{or } f(x) - f(a) &\approx f'(a)(x-a) \\
\text{or } f(x) - f(a) - m(x-a) &\approx 0
\end{aligned}
$$

$$\lim_{x \to a} \frac{f(x) - f(a) - m(x-a)}{|x-a|} = 0$$

Generalizing for $\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|} = \mathbf{0}$$

for some $m \times n$ matrix M.

$\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$ is **differentiable** at $\mathbf{a}$ if there exists an $m \times n$ matrix M such that

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}$$

Special Case: $m = 1, n = 2, M$ is 1×2 matrix$\nabla f = (f_x, f_y)$.

Example: $f(x, y) = x^2 + 2xy - y^2$ at (-1,2)

$$f(-1, 2) = -7$$

$$f_x(x, y) = 2x + 2y \text{ so } f_x(-1, 2) = 2$$

$$f_y(x, y) = 2x - 2y \text{ so } f_y(-1, 2) = -6$$

$$\nabla f(-1, 2) = (2, -6)$$

Equation of Tangent Plane:

$$\begin{aligned} z &= -7 + (2, -6) \cdot (x + 1, y - 2) \\ &= -7 + 2x + 2 - 6y + 12 \\ &= +7 + 2x - 6y \end{aligned}$$

Review meaning of $f_x(-1,2) = 2$ and $f_y(-1,2) = 6$

What is rate of change of f at (-1,2) if we approach along direction given by $\mathbf{v} = (3,4)$?

$$
\begin{aligned}
f_{\mathbf{v}}(-1,2) &= \lim_{t\to 0} \frac{f(-1+3t, 2+4t) - f(-1,2)}{t} \\
&= \lim_{t\to 0} \frac{(-1+3t)^2 + 2(-1+3t)(2+4t) - (2+4t)^2 - (-7)}{t} \\
&= \lim_{t\to 0} \frac{17t^2 - 18t}{t} \\
&= \lim_{t\to 0} (17t - 18) = -18
\end{aligned}
$$

Note: $(\nabla f) \cdot \mathbf{v} = (2,-6) \cdot (3,4) = (2)(3) + (-6)(4) = 6 - 24 - 18$

COINCIDENCE?

Major Theorems

If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$,

If all partial derivatives of $\mathbf{f}$ are continuous in a neighborhood of $\mathbf{a}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$.

If $\mathbf{f}$ is differentiable at $\mathbf{a}$,, then M is the matrix of first order partial derivatives.

Partial With Respect to a Vector

Let $f(x,y) = x^2y$ and $\mathbf{a} = (3,9)$ so $f(3,9) = 81$.
Find the partial derivative of f at (3,9) if we approach (3,9) along arbitrary vector $\mathbf{v} = (v_1, v_2)$.

We want $f_{\mathbf{v}}(\mathbf{a}) = \lim_{t \to 0} \dfrac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}$

$$
\begin{aligned}
f_{\mathbf{v}}(\mathbf{a}) &= \lim_{t \to 0} \frac{f(3 + tv_1, 9 + tv_2) - f(3,9)}{t} \\
&= \lim_{t \to 0} \frac{(3 + tv_1)^2(9 + tv_2) - (3^2)(9)}{t} \\
&= \lim_{t \to 0} \frac{(3^2 + 6tv_1 + t^2v_1^2)(9 + tv_2) - (3^2 099)}{t} \\
&= \lim_{t \to 0} \frac{(3^2)(9) + 3^2tv_2 + 6tv_1(9) + 6t^2v_1v_2 + t^2v_1^2(9) + t^3v_1^2v_2 - (3}{t} \\
&= \lim_{t \to 0} \left(3^2v_2 + 54v_1 + t6v_1v_2 + tv_1^2(9) + t^2v_1^2v_2\right) \\
&= 9v_2 + 54v_1 = 54v_1 + 9v_2 = (54, 9) \cdot (v_1, v_2)
\end{aligned}
$$