MATH 223

Hints and Answers for Assignment 18

Exercises 20, 21, 22, 23, and 24 in Chapter 5

20. When $x = y = 2$ or $x = y = -2$ we get the maximum; when $x = 2, y = -2$ or $x = -2, y = 2$, we get the minimum.

21: Form the function

$$F(x, y, \lambda) = x^\alpha y^\beta + \lambda(x + \frac{y}{2} - D)$$

Taking the gradient and setting the components equal to zero gives us three equations

$$\begin{aligned}
&(1)\ F_x = 0 : \alpha x^{\alpha-1} y^\beta + \lambda = 0 \\
&(2)\ F_y = 0 : \beta x^\alpha y^{\beta-1} + \frac{1}{2}\lambda = 0 \\
&(3)\ F_\lambda = 0 : x + \frac{y}{2} = D
\end{aligned}$$

If we multiply the second equation by 2, we find

$$\alpha x^{\alpha-1} y^\beta = -\lambda = 2\beta x^\alpha y^{\beta-1}.$$

Dividing through by $x^{\alpha-1}y^{\beta-1}$, we have

$$\alpha y = 2\beta x \text{ so } y = \frac{2\beta}{\alpha} x.$$

Use this equation for y in the equation $x + \frac{y}{2} = D$:

$$x + \frac{\beta}{\alpha} x = D \text{ so } x = \frac{\alpha}{\alpha+\beta} D,\ y = \frac{2\beta}{\alpha+\beta} D$$

For Zoey ($\alpha = \beta = 1/2$), the solution is $x = D/2,\ y = D$.
For Sydney $\alpha = 1/2, \beta = 1/5$), the solution is $x = 5D/7,\ y = 4D/7$.

22: To apply the method of Lagrange multipliers we combine the revenue function with the budget constraint to form a function of three variables with the form

$$F(x, y, \lambda) = 180x^{\frac{2}{3}} y^1 3 - \lambda(15x + 200y - 90000).$$

She should therefore pay for 4000 hours of labor and buy 150 tons of ingredients.

23: Anne's satisfaction function will be maximized and her time constraint will be met when $(x, y) = \left(\frac{2c}{3a}, \frac{c}{3b}\right)$.

24: Applying the method of Lagrange Multipliers to the constraint function $P(x, y)$ we have $F(x, y, \lambda) = 22x^{\frac{3}{4}} y^{\frac{1}{4}} - \lambda(ax + by - c)$.
Show $x = \frac{3c}{4a}$ and $\lambda = \frac{11}{2} 3^{\frac{3}{4}} a^{-\frac{3}{4}} b^{\frac{1}{4}}$.