MATH 223

Some Hints and Answers for Assignment 11

Exercises 17 and 18 in Chapter 4 and Problems A – C.

17. Show that the function f of one variable given by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } (x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable for all x but f' is not continuous at 0 so f is not continuously differentiable.

Hint: Wherever $x \neq 0$, $f(x)$ is the product and composition of differentiable functions, so it is differentiable.
Use Definition of Derivative to show

$$f'(0) = \lim_{h \to 0} h \sin \frac{1}{h}$$

and then show (give a proof) that this limit is 0.
The derivative of f to be

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} : x \neq 0 \\ 0 : x = 0 \end{cases} .$$

For $f'(x)$ to be continuous, the limit of $f'(x)$ as x approaches 0 must be zero. That is,

$$\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

must equal 0; Why does this limit not exist?

18: Replace x^2 with x^3 in the previous exercise and determine if the resulting function is continuously differentiable everywhere.

Problem A: For each of these functions f find gradient $\nabla f(\mathbf{x})$ of f at a general point in the domain of f:
(1) $f(x, y) = 2x^3 - 3y^2$: $\nabla f(x, y) = (f_x(x, y), f_y(x, y)) = (4x^2, -6y)$

(2) $f(x, y, z) = (5x - 7y)z = 5xz - 7yz$: $\nabla f(x, y, z) = (5z, -7z, 5x - 7y)$

(3) $f(x_1, x_2, x_3) = \frac{x_1 x_3}{x_2}$: $\nabla f(x_1, x_2, x_3) = \left(\frac{x_3}{x_2}, -\frac{x_1 x_3}{x_2^2}, \frac{x_1}{x_2} \right)$

Problem B: Write an equation in terms of the coordinate variables (x, y, z) for the tangent hyperplane for $f(x, y, z) = 2x^2 - y^2 + 3z^2$ when $x = y = z = 1$.

Solution: $w = 4 + 4(x - 1) - 2(y - 1) + 6(z - 1)$

Problem C: Let f be the real-valued function $f : \mathbb{R}^p \to \mathbb{R}^1$ defined by $f(\mathbf{x}) = |\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^p$. If $p = 2$, prove that $\nabla f(\mathbf{x}) = 2\mathbf{x}$. Is this result true for other values of p?

Note: For $p = 2$, $f(\mathbf{x}) = f(x, y) = = x^2 + y^2$ so that $f_x(x, y) = 2x$