MATH 223 Multivariable Calculus Examination 3 *December 1, 2021*

1. Let D be the circular disk in the plane made up of all points which are less than or equal to 2 units from the origin. Let f be the real-valued function defined by $f(x,y) = 5 - x^2 - y^2$

(a) Express $\int_D f$ as an iterated integral using standard (x,y) coordinates. Do **not** carry out the integration.

(b) Express $\int_D f$ as an integrated integral using polar coordinates. Do **not** evaluate the integral.

(c) Let $\mathbf{F} = \nabla f$ and use the definition of line integral to compute $\int_\gamma \mathbf{F}$ where γ is the portion of the boundary of D which runs from (2,0) to (0,2).

(d) Explain why the work done by **F** in moving an object along a path from (2,0) to (0,2) is independent of the path connecting them.

2. Several avid Boston Red Sox fans decide to build a monument to their favorite team. The monument will be in the shape of a column with circular cross sections varying linearly from diameter 12 inches at the base to a diameter of 8 inches at the top. The column will be 10 feet high. The density μ of the material in the column varies linearly along the length of the column from 50 pounds per cubic foot at the thick lower end to 40 pounds per cubic foot at the thin upper end. Set up an integral whose value is the total mass of the monument. (Note: pay careful attention to units of measurement). You do not need to evaluate the integral.

3. Let $f(x) = \int_0^1 \frac{u^x - 1}{\ln u}\, du$ for $x > -1$. (a) Use the Leibniz Rule to find $f'(x) = \frac{d}{dx} f$

(b) Integrate your answer for (a) with respect to x to obtain a simple formula for $f(x)$.

4. Let R_{uv} be the rectangle in the (u,v) – plane with vertices at (0,0), (1,0), (1,1) and (0,1). The transformation from R_{uv} to the (x,y)-plane be defined by $x = 2u + \frac{1}{v+1}, \quad y = u + v$

maps R_{uv} onto a region R_{xy} in the (x,y)-plane. Sketch R_{xy} and use Jacobi's Theorem to find its area. Check that all the hypotheses of the theorem apply.

5. (a) Find the value of the constant k so that the function $r(x,y) = ke^{-5x-4y}$ is a probability density function on ***Q***, the first quadrant of the plane.

(b) Using the probability density function in (a), suppose a point is selected at random from the first quadrant Q. What is the probability that the sum of its coordinates exceeds 3?

6. (a) Find the length of the curve in $\mathbb{R}^3$ which has parametrization $g(t) = (3t^2, 4t^3, -3t^4)$ for $0 \le t \le 2$.

(b) Let **F** be a continuously differentiable vector field on $\mathbb{R}^n$ and suppose $g: [0,1] \to \mathbb{R}^n$ parameterizes a curve γ which is *a flow line* for **F**. The function g describes motion along γ, giving directly the position of an object on γ for each time t. Let $s(t)$ be the speed associated with this motion.

(i) How is $s(t)$ computed from $g(t)$? (ii) What is the relationship between the line integral of **F** over γ and the integral $\int_0^1 s^2(t)\, dt$? Explain.

7. We have used the equality of mixed partials many times in our study of multivariable calculus but have never seen a proof of its validity if the function has certain properties. We will remedy this deficiency in this problem.

Suppose then that f is a real-valued function of two variables and f_x, f_y, f_{xy}, and f_{yx} are all continuous for all values of x and y.

(a) Explain why the equation is valid.

(b) Prove $f_{xy} = f_{yx}$ at all points of the plane using Leibniz's Rule.

Hint: Start by applying Leibniz's Rule to the equation in (a) and then differentiate both sides with respect to x.