

MATH 223 Fall 2023
Assignment 13
Due: Monday, October 16

Reading

Read carefully Section 4.6.1 “Economic Growth” and Section 5.1 “Differentiating Compositions of Functions” in our text *Multivariable Calculus: A Linear Algebra Based Approach*.

Writing

Write out solutions of Exercises 30, 31, 32 in Chapter 4 (Old 23, 24, 25) and Exercises A and B below.

- A. Show that the Mean Value formula of Theorem 4.4.1 (*Generalized Mean Value Theorem*) also takes the form

$$\frac{f(\mathbf{y})-f(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} = f_{\mathbf{u}}(\mathbf{x}_0)$$

where the unit vector is $\mathbf{u} = (\mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$.

- B. The Mean Value Theorem may fail for vector-valued functions. Consider the function

$$f(t) = (\sin t, \sin 2t), 0 \leq t \leq \pi.$$

Show that there is no point x_0 between x and y at which

$$\frac{f(y)-f(x)}{|y-x|} = f'(x_0) \text{ where } x = 0 \text{ and } y = \pi.$$

Here is an optional, extra credit problem:

- C. The fundamental derivative approximation $f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})\mathbf{h}$ for differentiable real-valued functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is the first degree portion of higher degree approximations. We define the *Nth-degree Taylor Approximation* to $f(\mathbf{x} + \mathbf{h})$ by

$$f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})\mathbf{h} + \frac{1}{2!} f_{\mathbf{hh}}(\mathbf{x}) + \frac{1}{3!} f_{\mathbf{hhh}}(\mathbf{x}) + \cdots + \frac{1}{N!} f_{\mathbf{hhhh}\dots\mathbf{hh}}(\mathbf{x})$$

where we assume that the required Nth-order derivatives are continuous on an open neighborhood of \mathbf{x} .

We define the higher-order derivatives with respect to the vector \mathbf{h} recursively by generalizing ordinary higher-order partials in this way:

$$f_{\mathbf{hh}}(\mathbf{x}) = \frac{\partial^2 f}{\partial \mathbf{h}^2}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{h}} \left(\frac{\partial f}{\partial \mathbf{h}} \right) (\mathbf{x})$$

$$f_{\mathbf{hhh}}(\mathbf{x}) = \frac{\partial^3 f}{\partial \mathbf{h}^3}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{h}} \left(\frac{\partial^2 f}{\partial \mathbf{h}^2} \right) (\mathbf{x})$$

...

$$f_{\mathbf{hhhh}\dots\mathbf{h}}(\mathbf{x}) = \frac{\partial^N f}{\partial \mathbf{h}^N}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{h}} \left(\frac{\partial^{N-1} f}{\partial \mathbf{h}^{N-1}} \right) (\mathbf{x})$$

Let $f(x, y) = \sin(x^2 + y)$, $\mathbf{x} = (0, 0)$, and $\mathbf{h} = (h, k)$. Compute the second-degree Taylor approximation to $f(h, k)$