<u>Proof</u> (of Global GB for surfaces w/o bdry) WTS: $\iint_S K\,dA = 2\pi\chi(S)$

Sps. $\mathcal{T}$ is a triangulation of S with each triangle Δ_i in a chart, $i = 1, 2, \ldots, F$ ← # faces.

Local GB applied to face Δ_i says:

exterior angles.

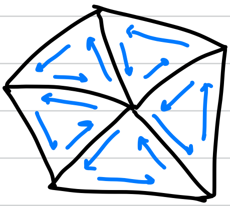
$$\int_{\bar{\alpha}_i} \kappa_g(t)\,dt + \iint_{\Delta_i} K\,dA = 2\pi - \sum_{i=0}^{2} \theta_i = \underbrace{\Phi_{i0} + \Phi_{i1} + \Phi_{i2}}_{\text{interior angles of ith triangle.}} - \pi$$

bdry of ith triangle.

So: $$\iint_S K\,dA = \sum_{i=1}^{F} \iint_{\Delta_i} K\,dA$$

$$= \sum_{i=1}^{F} \left(\Phi_{i0} + \Phi_{i1} + \Phi_{i2} - \pi \right) - \sum_{i=1}^{F} \int_{\bar{\alpha}_i} \kappa_g(t)\,dt$$

bdry of triangle



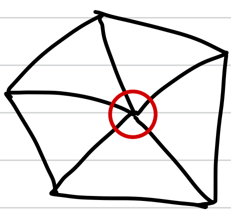
B/c S has no boundary, and b/c common edges are oppositely oriented,

$$\sum_{i=1}^{F} \int_{\bar{\alpha}_i} k_g(t)\,dt = 0.$$

contrib. from common edges cancel.

Thus
$$\iint_S K\,dA = \sum_{i=1}^{F} \left(\Phi_{i0} + \Phi_{i1} + \Phi_{i2} - \pi \right)$$

$$= \underbrace{\sum_{i=1}^{F} \left(\Phi_{i0} + \Phi_{i1} + \Phi_{i2} \right)}_{\text{sum of all interior angles of } \mathcal{P}\text{, organized by triangle.}} - \pi F$$



But: at each vertex, interior angles sum to 2π. Thus

$$\sum_{i=1}^{F} \left(\Phi_{i0} + \Phi_{i1} + \Phi_{i2} \right) = 2\pi V$$

So far: $\iint_S K\,dA = 2\pi \bar{V} - \pi \bar{F}$.

Finally, b/c we have no boundary, each face has 3 edges but each edge shares 2 faces. Thus

$$\bar{E} = \frac{3\bar{F}}{2} \rightsquigarrow 2\bar{E} = 3\bar{F}$$

So we have:

$$\begin{aligned}
\iint_S K\,dA &= 2\pi\bar{V} - \pi\bar{F} \qquad 0 \\
&= 2\pi\bar{V} - \pi\bar{F} + \pi(3\bar{F} - 2\bar{E}) \\
&= 2\pi\bar{V} + 2\pi\bar{F} - 2\pi\bar{E} \\
&= 2\pi(\bar{F} - \bar{E} + \bar{V}) \\
&= 2\pi\chi(S) \quad !
\end{aligned}$$

And finally, more generally:

Thm (Global Gauss-Bonnet)

If $R \subset S$ is a regular region with closed, simple, piecewise regular boundary $\bar{\alpha}$ such that each component $\bar{\alpha}_i$, $i = 1, \dots, n$ of ∂R is positively oriented relative to R and $\theta_1, \dots, \theta_n$ are the exterior angles, then

$$\sum_{i=1}^{n} \int_{\bar{\alpha}_i} \kappa_g(t)\,dt + \iint_R K\,dA + \sum_{i=1}^{n} \theta_i = 2\pi \chi(R).$$

↳ Note: for local version, R homeomorphic to disk, so $\chi(R) = 1$... consistent w/ local version.

proof: exercise.