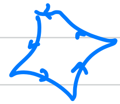


Proof of local Gauss-Bonnet Theorem:

WTS: $$\sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \kappa_g(t)\,dt + \iint_R K\,dA + \sum_{i=0}^{n} \theta_i = 2\pi$$

From Lemma 1, assume region R is contained in chart $(\bar{x}, U)$ with $F=0$.

Have: (Lemma 3) $$\kappa_g(s) = \frac{1}{2\sqrt{EG}}\left[G_u \frac{dv}{ds} - E_v \frac{du}{ds}\right] + \frac{d\varphi_i}{ds}$$

curve has pos. orientation rel. to R.

and: (Turning Tangents) $$\sum_{i=0}^{n} \left(\varphi_i(t_{i+1}) - \varphi_i(t_i)\right) + \sum_{i=0}^{n} \theta_i = +2\pi$$

So:

$$\sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \kappa_g(t)\,dt$$

$\bar{\alpha}_i(t) = (u(t), v(t))$

$\bar{x}^{-1} \circ \bar{\alpha}_i(t)$

Lemma 3

$$= \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \left(\frac{1}{2\sqrt{EG}}\left[G_u \frac{dv}{dt} - E_v \frac{du}{dt}\right]\right) dt + \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \frac{d\varphi_i}{ds}\,dt$$

want this to be $-\iint_R K\,dA$

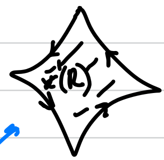
PTC $\Rightarrow \sum_{i=0}^{n} \varphi_i(t_{i+1}) - \varphi_i(t_i)$

$$= 2\pi - \sum_{i=0}^{n} \theta_i$$

all happening in uv-plane

Focus on $\star$:
$$\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}}\Big(\underbrace{\frac{G_u}{2\sqrt{EG}}\frac{dv}{dt}}_{Q\,dv}-\underbrace{\frac{E_v}{2\sqrt{EG}}\frac{du}{dt}}_{P\,du}\Big)\,dt$$

Recall Green's Thm:



$\bar{\alpha}$ = bdry curve

$$\int_{\bar{\alpha}} P\,du + Q\,dv = \iint_R \left(\frac{\partial Q}{\partial u}-\frac{\partial P}{\partial v}\right) du\,dv$$

By Green's Thm, $\star$ transforms to :

$$\iint_{\bar{x}^{-1}(R)}\Bigg(\underbrace{\frac{\partial}{\partial v}\left(\frac{E_v}{2\sqrt{EG}}\right)}_{-\frac{\partial P}{\partial v}}+\underbrace{\frac{\partial}{\partial u}\left(\frac{G_u}{2\sqrt{EG}}\right)}_{+\frac{\partial Q}{\partial u}}\Bigg)\,du\,dv$$

But finally, using defn of integral on region $R\subset S$,

$\star$ transforms one more time to :

$$\hookrightarrow \iint_R f\,dA = \iint_{\bar{x}^{-1}(R)} f(\bar{x}(u,v))\sqrt{EG-F^2}\,du\,dv$$

$$-\iint_{\bar{x}^{-1}(R)}\underbrace{-\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)-\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right)}_{\text{By lemma 2: this is } K.\ (!)}\sqrt{EG}\,du\,dv = -\iint_R K\,dA.$$

and we're done!