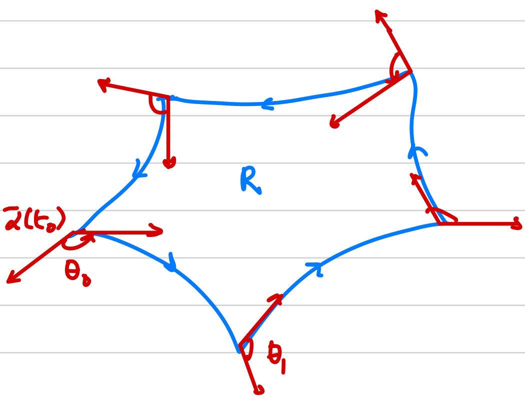
The Local Gauss-Bonnet Theorem

Thm (Local Gauss-Bonnet)

suppose R is completely contained in image $\bar{x}(U)$ of a chart.

Sps R is a simply-connected region in a regular surface S bounded by a simple, closed, piecewise regular curve $\bar{\alpha}$ which has positive orientation relative to R.

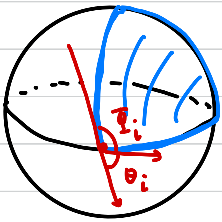
Let $\bar{\alpha}(t_0), \ldots, \bar{\alpha}(t_n)$ be the vertices of $\bar{\alpha}$, with exterior angles $\theta_0, \theta_1, \ldots, \theta_n$.



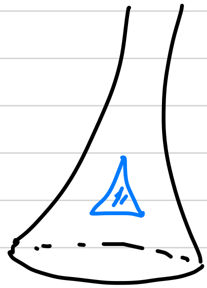
Then $$\underbrace{\sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \kappa_g(t)\,dt}_{\text{curves (edges)}} + \underbrace{\iint_R K\,dA}_{\text{surfaces (faces)}} + \underbrace{\sum_{i=0}^{n} \theta_i}_{\text{points (vertices)}} = 2\pi$$

Sps $\bar{\alpha}$ is a geodesic triangle on a surface S.

e.g.

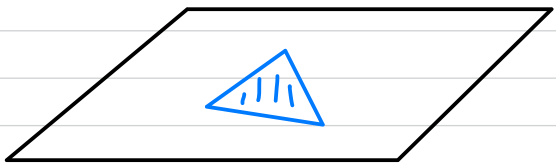


$K \equiv 1$



pseudosphere

$K \equiv -1$



$K \equiv 0$

The interior angles Φ_i are given by $\pi - \theta_i$

Since arcs of $\bar{\alpha}$ are geodesics, $\int_{t_i}^{t_{i+1}} \kappa_g(t)\,dt = 0 \quad \forall i.$

Thus we have

exterior angles

$$\iint_R K\,dA = 2\pi - \theta_0 - \theta_1 - \theta_2$$

$$= \Phi_0 + \Phi_1 + \Phi_2 - \pi$$

interior angles

* Corollary of local G-B *

i.e. $\quad \text{angle sum} = \pi + \underbrace{\iint_R K\,dA}_{\text{angle defect}}$

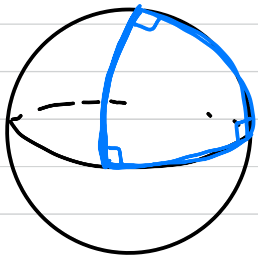
So:

1. For a plane ($K \equiv 0$), sum of interior angles of a triangle is π.

2. For a unit sphere ($K \equiv 1$), sum of interior angles of a triangle is $> \pi$ and $\underbrace{\text{angle excess}}_{\text{i.e. amount greater than } \pi}$ given by $\iint_{\text{triangle}} \overset{K}{1}\,dA = \text{area triangle}$

e.g.

area sphere radius r: $4\pi r^2$



$\text{area} = \frac{4\pi}{8} = \frac{\pi}{2}$ ← angle defect

$\text{angle sum} = \pi + \frac{\pi}{2}$

3. For a pseudosphere ($K \equiv -1$), sum of interior angles of a triangle is $< \pi$ and $\underbrace{\text{angle defect}}_{\text{i.e. amount less than } \pi}$ is given by $\iint_{\text{triangle}} \overset{K}{-1}\,dA = -(\text{area triangle})$