Step 2 Show K depends only on Γ_{ij}^{k}'s and E, F, G.

functions

(sketch)

Notice: $(\bar{x}_{uu})_v = (\bar{x}_{uv})_u$

basis $\{\bar{x}_u, \bar{x}_v, N\}$

we know $x_{uu} = \Gamma_{11}^{1}\bar{x}_u + \Gamma_{11}^{2}\bar{x}_v + eN$

$x_{uv} = \Gamma_{12}^{1}\bar{x}_u + \Gamma_{12}^{2}\bar{x}_v + fN$

So: take deriv both sides (left w.r.t. v, right w.r.t. u) to get:

$(\bar{x}_{uu})_v$

$$\Gamma_{11}^{1}\bar{x}_{uv} + \Gamma_{11}^{2}\bar{x}_{vv} + eN_v + (\Gamma_{11}^{1})_v\bar{x}_u + (\Gamma_{11}^{2})_v\bar{x}_v + e_vN$$

$$= \Gamma_{12}^{1}\bar{x}_{uu} + \Gamma_{12}^{2}\bar{x}_{vu} + fN_u + (\Gamma_{12}^{1})_u\bar{x}_u + (\Gamma_{12}^{2})\bar{x}_v + f_uN$$

$(\bar{x}_{uv})_u$

basis $\{\bar{x}_u, \bar{x}_v, N\}$

$$* \begin{cases} \Gamma_{11}^1 \underset{\sim}{\bar{x}_{uv}} + \Gamma_{11}^2 \underset{\sim}{\bar{x}_{vv}} + e \underset{\sim}{N_v} + (\Gamma_{11}^1)_v \bar{x}_u + (\Gamma_{11}^2)_v \bar{x}_v + e_v N \\ \quad = \Gamma_{12}^1 \underset{\sim}{\bar{x}_{uu}} + \Gamma_{12}^2 \underset{\sim}{\bar{x}_{vu}} + f \underset{\sim}{N_u} + (\Gamma_{12}^1)_u \bar{x}_u + (\Gamma_{12}^2) \bar{x}_v + f_u N \end{cases}$$

Now: replace $\bar{x}_{uv}$ and $\bar{x}_{uu}$ with respective Christoffel symbol expansions and replace N_u with $a_{11} \bar{x}_u + a_{21} \bar{x}_v$, N_v with $a_{12} \bar{x}_u + a_{22} \bar{x}_v$ ... so everything is in terms of basis $\{x_u, \bar{x}_v, N\}$.

unique expression of vectors.

Collect up like terms....

Equation * becomes: (sketch!)

P, Q, R, X, Y, Z each a big comb. of Γ_{ij}^k, a_{ij}, e, f

$$(\quad P \quad)\bar{x}_u + (\quad Q \quad)\bar{x}_v + (\quad R \quad)N$$

$$= (\quad X \quad)\bar{x}_u + (\quad Y \quad)\bar{x}_v + (\quad Z \quad)N$$

By uniqueness of expression, we have

Turns out:

$$\circledcirc \quad \left\{ \begin{aligned} Q &= \Gamma^1_{11}\Gamma^2_{21} + \Gamma^2_{11}\Gamma^2_{22} + e a_{22} + (\Gamma^2_{11})_v \\ Y &= \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + f a_{21} + (\Gamma^2_{12})_u \end{aligned} \right\} \text{equal}$$

Recall: $$\underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{local expression of } dN_p} = -\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \quad \Big\} \text{ Weingarten equations}$$

So $$a_{22} = \frac{fF - gE}{EG - F^2} \qquad a_{21} = \frac{eF - fE}{EG - F^2}$$

In $\circledcirc$, put Γ^k_{ij}'s on one side and ea_{22} and fa_{21} on other, and notice that $fa_{21} - ea_{22} = E\Big(\underbrace{\frac{eg - f^2}{EG - F^2}}_{K}\Big) = EK$ to conclude:

$$K = \frac{f a_{21} - e a_{22}}{E} = \frac{(\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^2\Gamma_{22}^2 + \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2}{E}$$

the Gauss formula.

Conclusion! From Step 1, we know Γ_{ij}^k's depend only on functions: E, F, G (i.e. I_p)

Gaussian curvature

Thus, from Step 2: we know K depends only on: Γ_{ij}^k's and $E, F, G, \ldots$ i.e. I_p

Therefore: K is an intrinsic invariant of S!

Gauss' Theorema Egregium

Corollary: If $\varphi : S_1 \to S_2$ is a local isometry, the $K_p = K_{\varphi(p)}$.