Step 1: Show Γ_{ij}^{k} depends only on the functions E, F, G.

Consider:

local expression of II_p

$$(\bar{x}_{uu} \cdot \bar{x}_u) = (\Gamma_{11}^{1} \bar{x}_u + \Gamma_{11}^{2} \bar{x}_v + eN) \cdot \bar{x}_u$$

$$= \Gamma_{11}^{1} E + \Gamma_{11}^{2} F$$

OTOH:

$$E_u = \frac{d}{du} E(u,v) = \frac{d}{du}(\bar{x}_u \cdot \bar{x}_u) = 2(\bar{x}_{uu} \cdot \bar{x}_u)$$

So: $\Gamma_{11}^{1} E + \Gamma_{11}^{2} F = \frac{1}{2} E_u$

Repeat for $\bar{x}_{uv}, \bar{x}_{vv}$ ....

... and get ...

$$\bar{x}_{uu}\cdot\bar{x}_u \longrightarrow \Gamma^1_{11}E + \Gamma^2_{11}F = \tfrac{1}{2}E_u$$

$$\bar{x}_{uu}\cdot\bar{x}_v \longrightarrow \Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \tfrac{1}{2}E_v$$

$$\begin{array}{ccc} ① & ② & ③ \\ \begin{bmatrix} E & F \\ F & G \end{bmatrix} & \begin{bmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{bmatrix} = & \begin{bmatrix} \tfrac{1}{2}E_u \\ F_u - \tfrac{1}{2}E_v \end{bmatrix} \end{array}$$

$$\bar{x}_{uv}\cdot\bar{x}_u \longrightarrow \Gamma^1_{12}E + \Gamma^2_{12}F = \tfrac{1}{2}E_v$$

$$\bar{x}_{uv}\cdot\bar{x}_v \longrightarrow \Gamma^1_{12}F + \Gamma^2_{12}G = \tfrac{1}{2}G_u$$

$$\begin{array}{ccc} ① & ② & ③ \\ \begin{bmatrix} E & F \\ F & G \end{bmatrix} & \begin{bmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{bmatrix} = & \begin{bmatrix} \tfrac{1}{2}E_v \\ \tfrac{1}{2}G_u \end{bmatrix} \end{array}$$

$$\bar{x}_{vv}\cdot\bar{x}_u \longrightarrow \Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \tfrac{1}{2}G_u$$

$$\bar{x}_{vv}\cdot\bar{x}_v \longrightarrow \Gamma^1_{22}F + \Gamma^2_{22}G = \tfrac{1}{2}G_v$$

$$\begin{array}{ccc} ① & ② & ③ \\ \begin{bmatrix} E & F \\ F & G \end{bmatrix} & \begin{bmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{bmatrix} = & \begin{bmatrix} F_v - \tfrac{1}{2}G_u \\ \tfrac{1}{2}G_v \end{bmatrix} \end{array}$$

②'s cover all Christoffel symbols.

① is invertible. ③'s depend only on E, F, G. as functions

So $② = ①^{-1}③$ gives Christoffel symbols completely in terms of functions E, F, G, ie. I_p.

need functions so can take derivatives

Step 1: