How do $e, f, g$ relate to $E, F, G$?

chart $\bar{x}(u,v)$
so $\{\bar{x}_u, \bar{x}_v\}$ basis for $T_pS$.

$dN_p(\bar{x}_u)$ $dN_p(\bar{x}_v)$

$N_u, N_v \in T_pS$

So
$$N_u = a_{11}\bar{x}_u + a_{21}\bar{x}_v$$
$$N_v = a_{12}\bar{x}_u + a_{22}\bar{x}_v$$
for some $a_{11}, a_{21}, a_{12}, a_{22}$

The matrix of $dN_p : T_pS \to T_pS$ relative to $\{\bar{x}_u, \bar{x}_v\}$ is:

$$\begin{bmatrix} | & | \\ dN_p(\bar{x}_u) & dN_p(\bar{x}_v) \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

mystery entries.

* local expression of $dN_p$

So: $$-e = (N_u \cdot \bar{x}_u)$$

$$= (a_{11}\bar{x}_u + a_{21}\bar{x}_v \cdot \bar{x}_u)$$

$$= a_{11}E + a_{21}F$$

$$-f = (\overset{a_{12}\bar{x}_u + a_{22}\bar{x}_v}{N_v} \cdot \bar{x}_u) = (\overset{a_{11}\bar{x}_u + a_{21}\bar{x}_v}{N_u} \cdot \bar{x}_v)$$

so $-f = a_{12}E + a_{22}F = a_{11}F + a_{21}G$

and $$-g = (\overset{a_{12}\bar{x}_u + a_{22}\bar{x}_v}{N_v} \cdot \bar{x}_v)$$

$$= a_{12}F + a_{22}G.$$

ie. $$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

all easy to compute

But then : $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \dfrac{-1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$

matrix of mystery components of $dN_p$

called Weingarten equations

$$= \frac{-1}{EG - F^2} \begin{bmatrix} Ge - Ff & Gf - Fg \\ -Fe + Ef & -Ff + Eg \end{bmatrix}$$