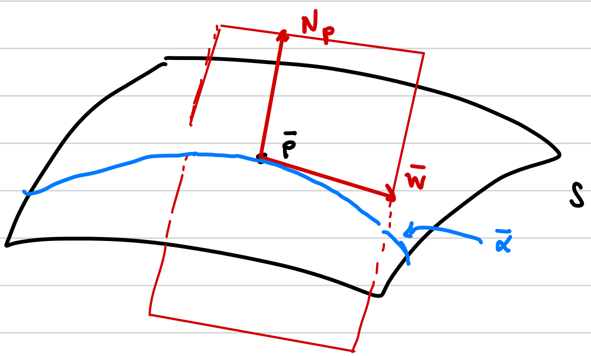
Since normal curvature depends on direction but not specific curve, sufficies to consider normal sections of the surface through $\bar{p}$.

Given unit vector $\bar{w} \in T_p S$, let C be the curve obtained by intersecting S w/ plane determined by N_p and $\bar{w}$.



C is called the normal section of S defined by $\bar{w}$.

Note: If $C = \bar{\alpha}(s)$ p.b.a.l., with $\bar{\alpha}(0) = \bar{p}$, then

$$\bar{\alpha}'(0) = \bar{w} \quad \text{and} \quad \bar{n}(0) = \pm N_p$$

(b/c C a plane curve)

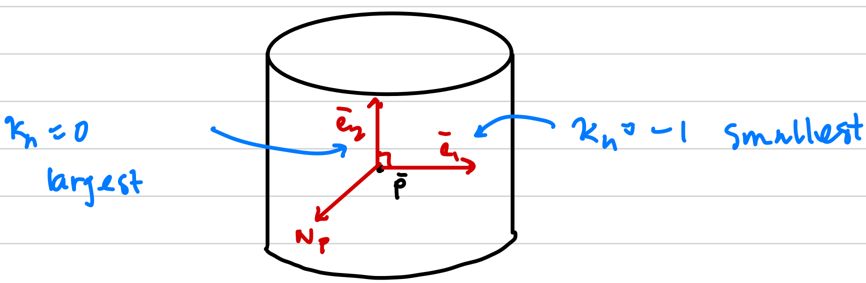
So $|\mathrm{II}_p(\bar{w})| = |\kappa_n| = \kappa$ ← curvature of $\bar{\alpha}$

Recall: dN_p is self-adjoint so there is an ONB $\{\bar{e}_1, \bar{e}_2\}$ of eigenvectors of dN_p and neg. eigenvalues are max/min values of

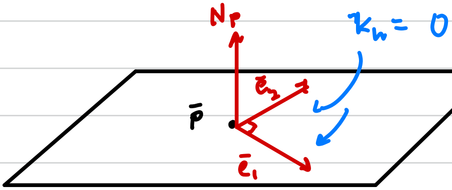
$$-\left(dN_p(\bar{w}) \cdot \bar{w}\right)_p = \mathrm{II}_p(\bar{w}) \text{ where } \bar{w} \text{ a unit vector.}$$

normal curvature

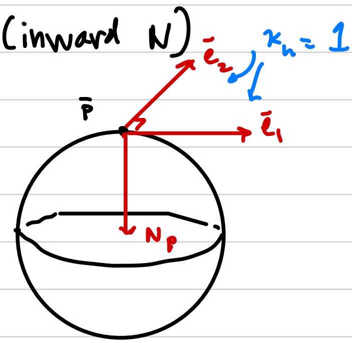
Ex cylinder (outward N)



Ex plane (upward N)

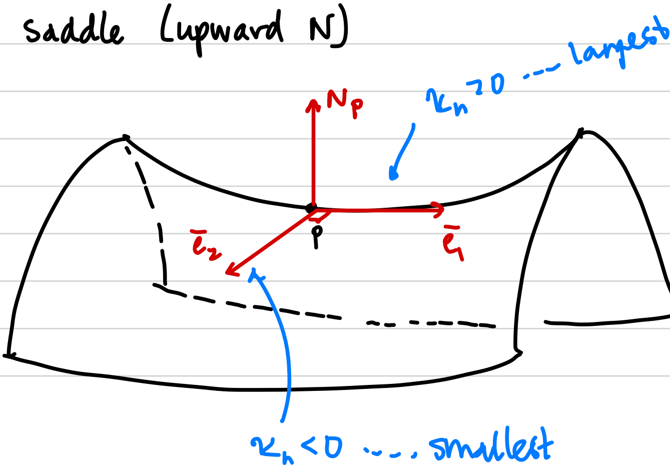


Ex unit sphere (inward N)



since eigenvalues are repeated, all vectors are eigenvectors, so any ONB is an eigenvector ONB.

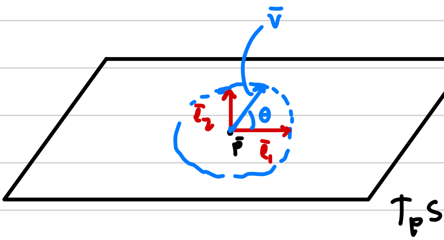
Ex saddle (upward N)



Defn The min/max normal curvatures κ_1 and κ_2 are the <u>principal curvatures</u> at $\bar p$. The corresponding directions given by the eigenvectors $\bar e_1$ and $\bar e_2$ are the <u>principal directions</u>.

Ex Sphere. principal directions = all direction

Note: if you know principal curvatures/directions, you can figure out normal curvature in all other directions.



unit vector $\bar v = \cos\theta\, \bar e_1 + \sin\theta\, \bar e_2$

So $\kappa_n(\bar v) = \mathrm{II}_p(\bar v) = -\left(dN_p(\bar v)\cdot \bar v\right)_p$

$dN_p(\bar e_1) = -\kappa_1 \bar e_1$
$dN_p(\bar e_2) = -\kappa_2 \bar e_2$

$$= -\Big(dN_p\big(\overbrace{\cos\theta\, \bar e_1 + \sin\theta\, \bar e_2}^{\bar v}\big)\cdot \overbrace{\cos\theta\, \bar e_1 + \sin\theta\, \bar e_2}^{\bar v}\Big)_p$$

$\kappa_n(\bar v)$ in terms of κ_1, κ_2 and principal directions

$$= (\kappa_1 \cos\theta\, \bar e_1 + \kappa_2 \sin\theta\, \bar e_2)\cdot(\cos\theta\, \bar e_1 + \sin\theta\, \bar e_2)_p$$

$$= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$