Thm If $N : S \to S^2$ is the Gauss map, then

$$dN_p : T_pS \to T_pS$$

is a self-adjoint linear map.

Corollary: Everything we know about self-adjoint maps on 2-dim'l vector spaces applies.

In particular, dN_p has an orthonormal basis of eigenvectors.

<u>proof</u> NTS: $dN_p(\bar{v}) \cdot \bar{w} = \bar{v} \cdot dN_p(\bar{w})$ for all $\bar{v}, \bar{w} \in T_pS$.

Sps. $(\bar{x}, U)$ is a chart about $\bar{p}$ and $\{\bar{x}_u, \bar{x}_v\}$ is the corresponding (local) basis of T_pS. If we can show

$$dN_p(\bar{x}_u) \cdot \bar{x}_v = \bar{x}_u \cdot dN_p(\bar{x}_v) \qquad *$$

we are done b/c

$$\Big[dN_p(\underbrace{a\bar{x}_u + b\bar{x}_v}_{\bar{v}})\Big] \cdot (\underbrace{c\bar{x}_u + d\bar{x}_v}_{\bar{w}})$$

relation holds automatically

apply * here

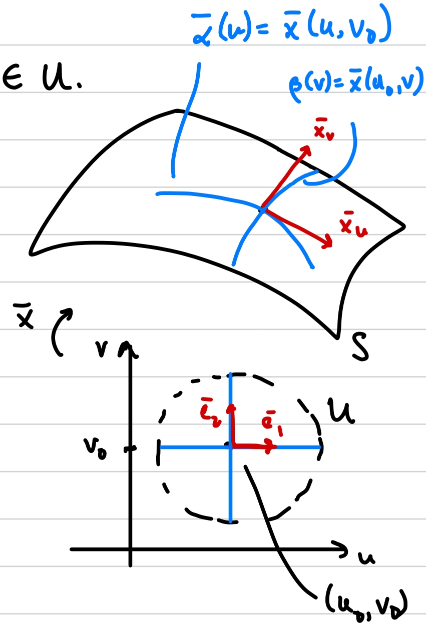
$$= ac\, dN_p(\bar{x}_u) \cdot \bar{x}_u + bc\, dN_p(\bar{x}_v) \cdot \bar{x}_u + ad\, dN_p(\bar{x}_u) \cdot \bar{x}_v + bd\, dN_p(\bar{x}_v) \cdot \bar{x}_v$$

$$= ac\, \bar{x}_u \cdot dN_p(\bar{x}_u) + bc\, \bar{x}_v \cdot dN_p(\bar{x}_u) + ad\, \bar{x}_u \cdot dN_p(\bar{x}_v) + bd\, \bar{x}_v \cdot dN_p(\bar{x}_v)$$

$$= [\underbrace{a\bar{x}_u + b\bar{x}_v}_{\bar{v}}] \cdot \Big[dN_p(\underbrace{c\bar{x}_u + d\bar{x}_v}_{\bar{w}})\Big]$$

Now, let's show $dN_p(\bar{x}_u)\cdot\bar{x}_v = \bar{x}_u \cdot dN_p(\bar{x}_v)$.

Sps $\bar{p} = \bar{x}(u_0, v_0)$ for $(u_0, v_0) \in U$.



Note that $\bar{x}_u = \frac{d}{du}(\bar{x}(u, v_0))$

and $\bar{x}_v = \frac{d}{dv}(\bar{x}(u_0, v))$

Let N_u denote $dN_p(\bar{x}_u)$.

$$\frac{d}{du}\Big|_{u_0}(N(\bar{x}(u, v_0)))$$

and N_v denote $dN_p(\bar{x}_v)$

$$\frac{d}{dv}\Big|_{v_0} N(\bar{x}(u_0, v))$$

Now, since $N(u,v) = \pm \dfrac{\bar{x}_u \times \bar{x}_v}{|\bar{x}_u \times \bar{x}_v|}$, we have

$$N \perp \bar{x}_u \qquad \text{and} \qquad N \perp \bar{x}_v$$

3-dim'l dot product

i.e. $\quad N \cdot \bar{x}_u = N \cdot \bar{x}_v = 0$ for all $(u,v) \in U$.

constant

Taking derivatives gives:

$$\frac{\partial}{\partial v}(N \cdot \bar{x}_u) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial u}(N \cdot \bar{x}_v) = 0$$

This implies:

$$N_v \cdot \bar{x}_u + N \cdot \bar{x}_{uv} = 0$$

and

$$N_u \cdot \bar{x}_v + N \cdot \bar{x}_{vu} = 0$$

equal.

equality of mixed partials

Since $\bar{x}_{uv} = \bar{x}_{vu}$, we conclude $N_u \cdot \bar{x}_v = N_v \cdot \bar{x}_u$

i.e. $\quad dN_p(\bar{x}_u) \cdot \bar{x}_v = \bar{x}_u \cdot dN_p(\bar{x}_v)$

so dN_p is self-adjoint. ✓