keep dN_p in mind

A linear algebra detour: self-adjointness.

e.g. dot product

Defn. Sps V is a vector space with inner product $\langle \, , \rangle$.

A linear transformation $A : V \to V$ is called **self-adjoint** if

$$\langle A\bar{v}, \bar{w} \rangle = \langle \bar{v}, A\bar{w} \rangle$$

for all $\bar{v}, \bar{w} \in V$.

Ex. $A : V \longrightarrow V$ $\quad \bar{x} \mapsto k\bar{x}$. scalar k

$$\langle k\bar{v}, \bar{w} \rangle = k\langle \bar{v}, \bar{w} \rangle = \langle \bar{v}, k\bar{w} \rangle$$

Self-adjoint maps have excellent properties.

keep Tps in mind.

Sps. V is 2-dim'l.

2-dim'l

1. Sps. $A: V \to V$ is self-adjoint. Then A has 2 real (possibly equal) eigenvalues ($A\bar{v} = \lambda \bar{v}$) and V has an orthonormal basis of eigenvectors of A.

eigenvalue eigenvector

follows from fact that if $\varphi: V \to V$ is s.a. and A is a matrix of φ relative to an orthonormal basis, then A is symmetric, i.e. $A^T = A$.

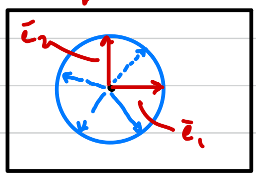
i.e.

2. If eigenvalues are equal, all vectors in V are eigenvectors, so any orthonormal pair is an ONB. If eigenvalues $\lambda_1 \neq \lambda_2$ we have an ONB $\{\bar{e}_1, \bar{e}_2\}$ with $A\bar{e}_1 = \lambda_1 \bar{e}_1$ and $A\bar{e}_2 = \lambda_2 \bar{e}_2$.

3. Restrict A to unit length vectors in V and consider the function

$$\bar{v} \longmapsto \underbrace{\langle A\bar{v}, \bar{v} \rangle}_{\text{real \#}}$$

$\langle A\bar{e}_2, \bar{e}_2 \rangle = \langle \lambda_2 \bar{e}_2, \bar{e}_2 \rangle$
$= \lambda_2 \langle \bar{e}_2, \bar{e}_2 \rangle$
$= \lambda_2$

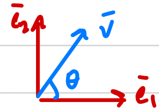


$\langle A\bar{e}_1, \bar{e}_1 \rangle = \langle \lambda_1 \bar{e}_1, \bar{e}_1 \rangle$
$= \lambda_1 \langle \bar{e}_1, \bar{e}_1 \rangle$
$= \lambda_1$

If $\lambda_1 > \lambda_2$ are eigenvalues then $\langle A\bar{v}, \bar{v} \rangle$ has

max value, of λ_1 in direction of: $\pm \bar{e}_1$

and min value, of λ_2 in direction of: $\pm \bar{e}_2$



4. B/c $\langle \, , \, \rangle$ is linear in both terms, if you know $\langle A\bar{e}_1, \bar{e}_1 \rangle$ and $\langle A\bar{e}_2, \bar{e}_2 \rangle$, you know $\langle A\bar{v}, \bar{v} \rangle$ for all $\bar{v} \in V$.

In particular, if $\bar{v}$ is a unit vector, $v = \cos\theta \bar{e}_1 + \sin\theta \bar{e}_2$

So $\langle A\bar{v}, \bar{v} \rangle = \langle \cos\theta A\bar{e}_1 + \sin\theta A\bar{e}_2 , \cos\theta \bar{e}_1 + \sin\theta \bar{e}_2 \rangle$

$= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$

$$\left\{ \begin{array}{l} A\bar{e}_1 = \lambda_1 \bar{e}_1 \\ A\bar{e}_2 = \lambda_2 \bar{e}_2 \\ \langle \bar{e}_1, \bar{e}_2 \rangle = 0 \end{array} \right.$$