Defn A map $f: S_1 \to S_2$ is a local isometry at $\bar{p}$ if there is an open set $V \subset S_1$ about $\bar{p}$ s.t. $f|_V$ is an isometry b/w V and $f(V)$.

(isometry $\Rightarrow$ local isometry
local isometry $\not\Rightarrow$ isometry ...
similar for diffeos)

doesn't necessarily cover all of S_1

Prop Sps $(\bar{x}, U)$ is a parametrization of S_1. Then $f: S_1 \to S_2$ is a local isometry at all points $\bar{p}$ in $\bar{x}(U)$ if and only if

$$E(\bar{p}) = df_p(\bar{x}_u) \cdot df_p(\bar{x}_u)$$

$$F(\bar{p}) = df_p(\bar{x}_u) \cdot df_p(\bar{x}_v)$$

$$G(\bar{p}) = df_p(\bar{x}_v) \cdot df_p(\bar{x}_v).$$

dot product preserved

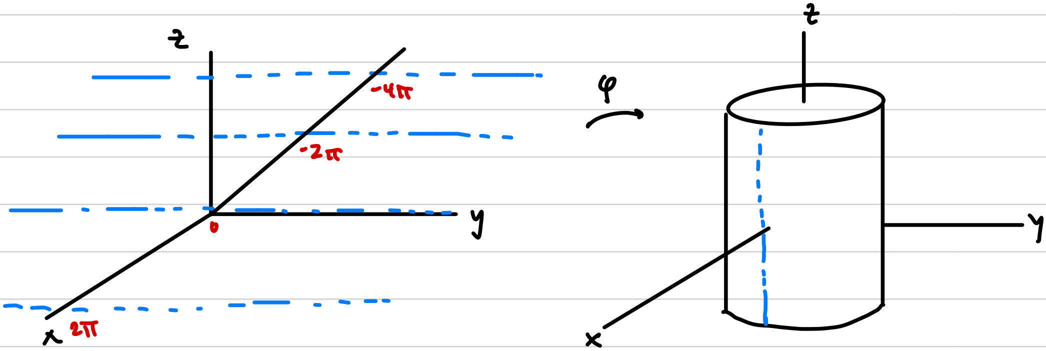
proof: follows directly from definition of local isometry and fact that dot product is linear ~~is linear~~ in both terms: $(c\bar{x}_u + d\bar{x}_v) \cdot (g\bar{x}_u + h\bar{x}_v)$

(exercise) $= cd\,\bar{x}_u \cdot \bar{x}_u + (ch + dg)\bar{x}_u \cdot \bar{x}_v + dh\,\bar{x}_v \cdot \bar{x}_v$

$= $ etc.

<u>Ex</u> xy-plane and cylinder are locally isometric.

$$\varphi(x,y,0) = (\cos x, \sin x, y)$$



Consider charts $\bar{x}(u,v) = (u,v,0)$, $0<u<2\pi$, $v\in\mathbb{R}$

and $\bar{y}(s,t) = (\cos s, \sin s, t)$, $0<s<2\pi$, $t\in\mathbb{R}$.

Notice that $\bar{y}^{-1}\circ\varphi\circ\bar{x}(u,v) = \bar{y}^{-1}(\varphi(u,v,0))$

$$= \bar{y}^{-1}(\cos u, \sin u, v)$$

$$= (u,v)$$

matrix representation of $d\varphi_p$, w/ bases $\{\bar{x}_u, \bar{x}_v\}$ and $\{\bar{y}_s, \bar{y}_t\}$.

Thus, at all points $\bar{p} = \bar{x}(u,v)$, $d\varphi_{\bar{p}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$1\bar{x}_u + 0\bar{x}_v$

So $\bar{x}_u = 1\bar{x}_u + 0\bar{x}_v$, thus $d\varphi_{\bar{p}}(\bar{x}_u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \bar{y}_s$

$1\bar{y}_s + 0\bar{y}_t$

Similarly, $d\bar{\varphi}_p(\bar{x}_v) = \bar{y}_t$

$\bar{x}(u,v) = (u,v,0)$

Note: $\bar{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\bar{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ so $E(\bar{p}) = 1$, $F(\bar{p}) = 0$, $G(\bar{p}) = 1$

$y(s,t) = (\cos s, \sin s, t)$

and $\bar{y}_s = \begin{bmatrix} -\sin s \\ \cos s \\ 0 \end{bmatrix}$, $\bar{y}_t = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so

$d\varphi_p(\bar{x}_u) \cdot d\varphi_p(\bar{x}_u) = 1$

$d\varphi_p(\bar{x}_u) \cdot d\varphi_p(\bar{x}_v) = 0$

$d\varphi_p(\bar{x}_v) \cdot d\varphi_p(\bar{x}_v) = 1$

$d\varphi_p(\bar{x}_u)$

$d\varphi_p(\bar{x}_v)$

Thus, by proposition, plane and cylinder are locally isometric.