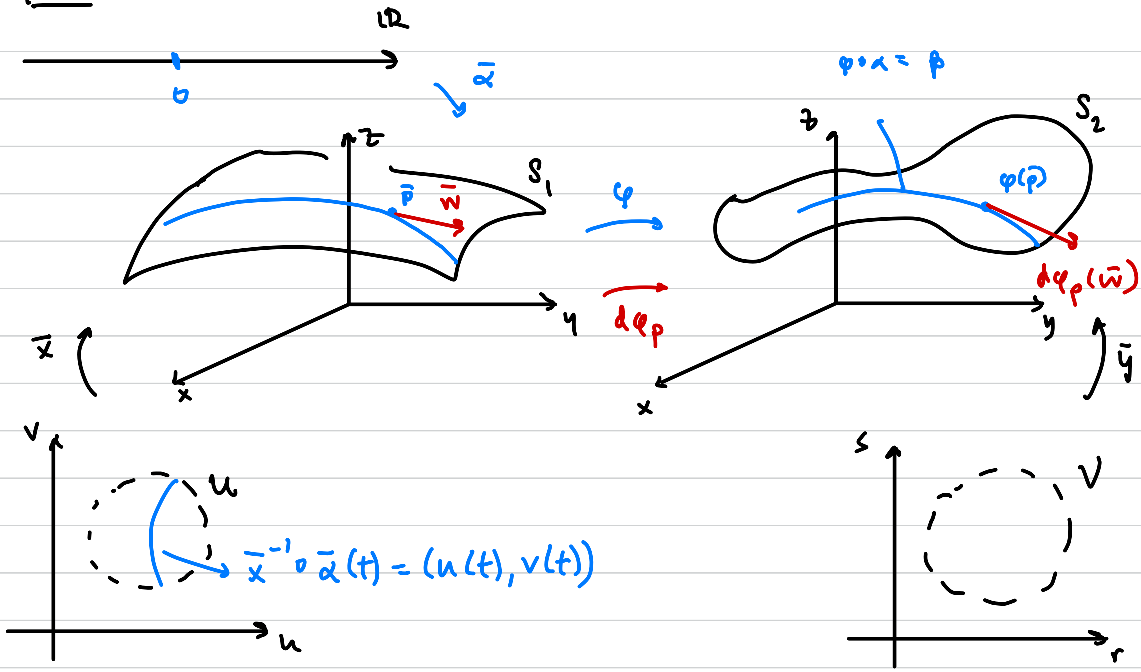
Thm Sps. $\varphi: S_1 \to S_2$. Sps $\bar{w} \in T_pS$ is given by $\bar{\alpha}'(0)$.

The map $d\varphi_p$ is linear and does not depend on choice of $\bar{\alpha}$.

*

Remark: Thm also holds for $h: S_1 \to \mathbb{R}$...

$dh_p: T_pS \to \mathbb{R}$ is linear and doesn't depend on choice of $\bar{\alpha}$. (exercise) proof



Let $(\bar{x}, U)$ and $(\bar{y}, V)$ be parametrizations about $\bar{p}$ and $\varphi(\bar{p})$. So

$\{\bar{x}_u, \bar{x}_v\}$ and $\{\bar{y}_r, \bar{y}_s\}$

are bases of $T_{\bar{p}} S_1$ and $T_{\varphi(\bar{p})} S_2$ respectively.

GOAL: Express $d\varphi_p$ in terms of $\{\bar{x}_u, \bar{x}_v\}$ and $\{\bar{y}_r, \bar{y}_s\}$ and show it is a linear map.

(In process, show $d\varphi_p(\bar{w})$ is independent of choice of $\bar{\alpha}$ used to define $\bar{w}$.)

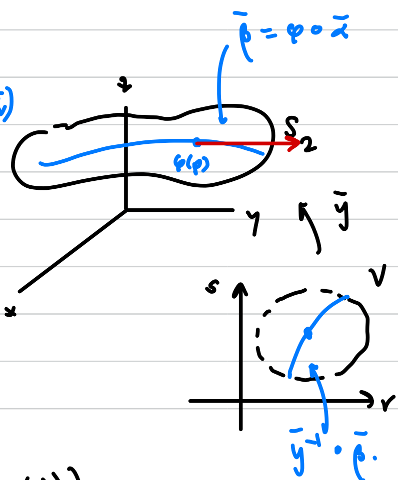
In local coordinates given by $\bar{x}$ and $\bar{y}$, suppose

$$\bar{y}^{-1} \circ \varphi \circ \bar{x}(u,v) = (\underbrace{\varphi_1(u,v)}_{r}, \underbrace{\varphi_2(u,v)}_{s})$$

and $\bar{x}^{-1} \circ \bar{\alpha}(t) = (u(t), v(t))$.

(Recall: $\bar{\alpha}'(0) = \bar{w}$

so $\bar{w} = u'(0)\bar{x}_u + v'(0)\bar{x}_v$)

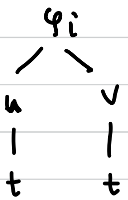


Let $\bar{\beta} = \varphi \circ \bar{\alpha}(t)$. (Recall $\bar{\beta}'(0) = d\varphi_p(\bar{w})$)

So
$$\begin{aligned}
\bar{y}^{-1} \circ \bar{\beta}(t) &= \bar{y}^{-1} \circ \varphi \circ \bar{\alpha}(t) \\
&= (\bar{y}^{-1} \circ \varphi \circ \bar{x}) \circ (\bar{x}^{-1} \circ \bar{\alpha})(t) \\
&= (\varphi_1(u(t), v(t)), \varphi_2(u(t), v(t)))
\end{aligned}$$

deriv w.r.t. t

$$(\bar{y}^{-1} \circ \bar{\beta})'(0) = \left(\frac{d\varphi_1}{dt}(0), \ \frac{d\varphi_2}{dt}(0) \right)$$



new notation

$$= \frac{d\varphi_1}{dt}(0)\,\bar{e}_1 + \frac{d\varphi_2}{dt}(0)\,\bar{e}_2$$

$$= \underbrace{\left[\frac{\partial \varphi_1}{\partial u} u'(0) + \frac{\partial \varphi_1}{\partial v} v'(0) \right]}_{*} \bar{e}_1 + \underbrace{\left[\frac{\partial \varphi_2}{\partial u} u'(0) + \frac{\partial \varphi_2}{\partial v} v'(0) \right]}_{**} \bar{e}_2$$

But $\bar{\beta}'(0)$ has same relationship to basis $\{\bar{y}_r, \bar{y}_s\}$ that $(\bar{y}^{-1} \circ \bar{\beta})'(0)$ has to $\{\bar{e}_1, \bar{e}_2\}$. (Similar for $\bar{\alpha}'(0)$ with basis $\{\bar{x}_u, \bar{x}_v\}$.)

So: $\bar{w} = \bar{\alpha}'(0)$ in terms of basis $\{\bar{x}_u, \bar{x}_v\}$ is: $\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}$ and

$$d\bar{\varphi}_p(\bar{w}) = \bar{\beta}'(0) = \begin{bmatrix} * \\ ** \end{bmatrix}$$

$d\varphi_p(\bar{w})$ in terms of basis $\{\bar{y}_r, \bar{y}_s\}$.

$$= \underbrace{\begin{bmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{bmatrix}}_{\text{matrix of } d\varphi_p \text{ in terms of } \{\bar{x}_u, \bar{x}_v\} \text{ and } \{\bar{y}_r, \bar{y}_s\}} \underbrace{\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}}_{\bar{\alpha}'(0)}$$

2-dim'l vector spaces!

Since $d\bar{\varphi}_p : T_p S_1 \to T_{\varphi(p)} S_2$ can be expressed

in terms of a matrix, it is a linear map.

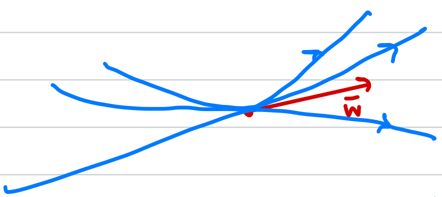
Further: $d\varphi_p(\bar{w})$ depends only on coords of $\bar{w}$

w.r.t. $\{\bar{x}_u, \bar{x}_v\}$, namely $u'(0)$ and $v'(0)$

scalars

Thus $d\varphi_p(\bar{w})$ doesn't depend on choice of $\bar{\alpha}$ that

represents $\bar{w}$.



many $\bar{\alpha}$'s give same $\bar{w}$.

Note: matrix of $d\varphi_p$ is a 2×2 matrix b/c it

maps a 2-dim'l vector space T_pS_1 to a 2-dim'l

vector space $T_{\varphi(p)} S_2$.