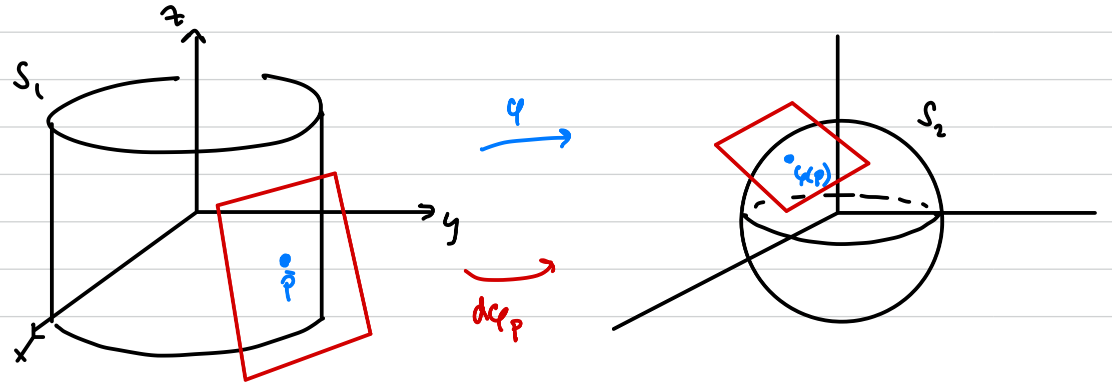
The Derivative (Differential) of a Smooth Map

Sps. $f : \mathbb{R}^n \to \mathbb{R}^m$ is diff'ble at $\bar{p}$. We know the (total) derivative $df_{\bar{p}} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map approximating f near $\bar{p}$.

Now, if $\varphi : S_1 \to S_2$ is diff'ble at $\bar{p}$ (i.e. $\bar{y}^{-1} \circ \varphi \circ \bar{x}$ is a diff'ble map $\mathbb{R}^2 \to \mathbb{R}^2$) we can define the derivative of φ at $\bar{p}$:

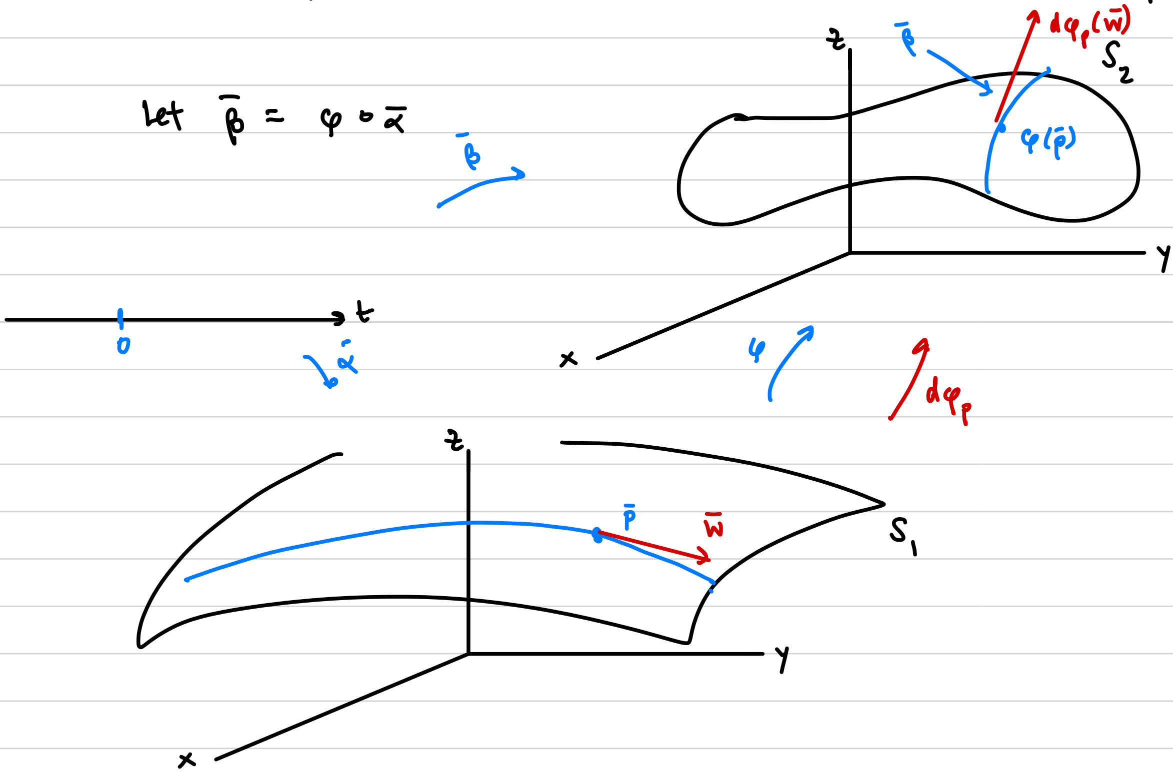
$$d\varphi_p : T_p S_1 \longrightarrow T_{\varphi(p)} S_2$$



To define $d\varphi_p$:

Let $\bar{w} \in T_pS$, so $\bar{w} = \bar{\alpha}'(0)$ where $\bar{\alpha}(t) \in S_1$ for all t, $\bar{\alpha}(0) = \bar{p}$

Let $\bar{\beta} = \varphi \circ \bar{\alpha}$



Then $\bar{\beta}(t) \in S_2$ and $\bar{\beta}(0) = \varphi(\bar{p})$.

<u>Defn</u> The <u>derivative</u> $d\varphi_{\bar{p}}$ of φ at $\bar{p}$ is the map

$$d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$$

given by

$$d\varphi_p(\bar{w}) = \beta'(0) = (\varphi \circ \bar{\alpha})'(0)$$

tan space to $\mathbb{R}$ at $h(p)$.

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Defn If $h: S \to \mathbb{R}$, the derivative $dh_p : T_pS \to \mathbb{R}$

is given by

$$dh_p(\bar{w}) = (h \circ \bar{\alpha})'(0)$$

Let $\bar{\alpha}$ be such that:

$\bar{\alpha}(t) \in S$

$\bar{\alpha}(0) = p$

$\alpha'(0) = \bar{w}$