Smooth Maps on Surfaces

(usually $\mathbb{R}$)

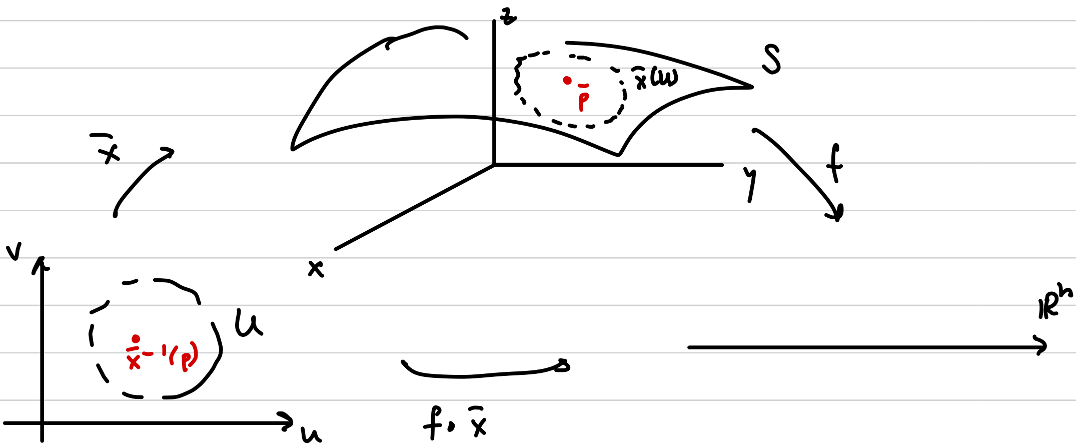
Define in two steps: 1. $f: S \to \mathbb{R}^n$ diffble.

2. $f: S_1 \to S_2$ diffble

any n

__Defn__ A function $f: S \to \mathbb{R}^n$ is __differentiable at $\bar{p}$__ if there exists a chart $(\bar{x}, U)$ such that $\bar{p} \in \bar{x}(U)$ and $f \circ \bar{x}$ is diffble at $\bar{x}^{-1}(p)$

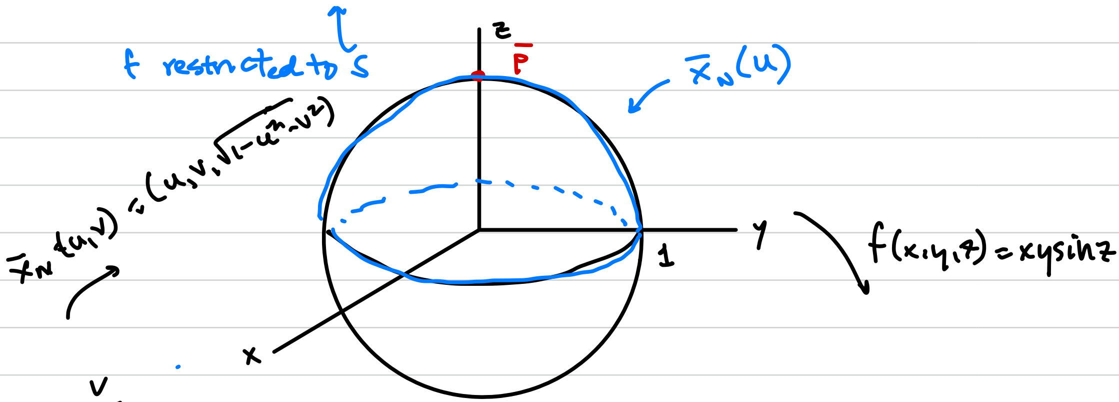
If f is diffble at all $\bar{p} \in S$, we say f is __differentiable__.

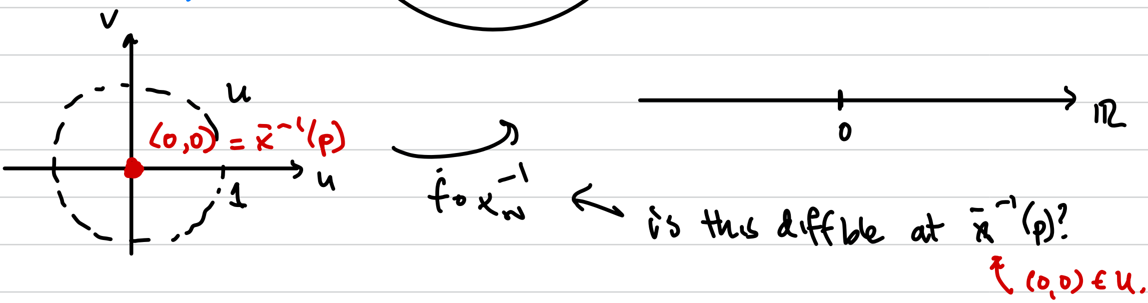


(using fact that we know what $f: \mathbb{R}^n \to \mathbb{R}^m$ diffble means ~ generalizing)

Ex. S = sphere $\quad f(x,y,z) = xy\sin z \quad \bar{p} = (0,0,1)$

Show $f\big|_S$ is diff'ble at $\bar{p}$.





Check:

$f \circ \bar{x}_N : U \subset \mathbb{R}^2 \to \mathbb{R}$ $\qquad u^2+v^2<1$

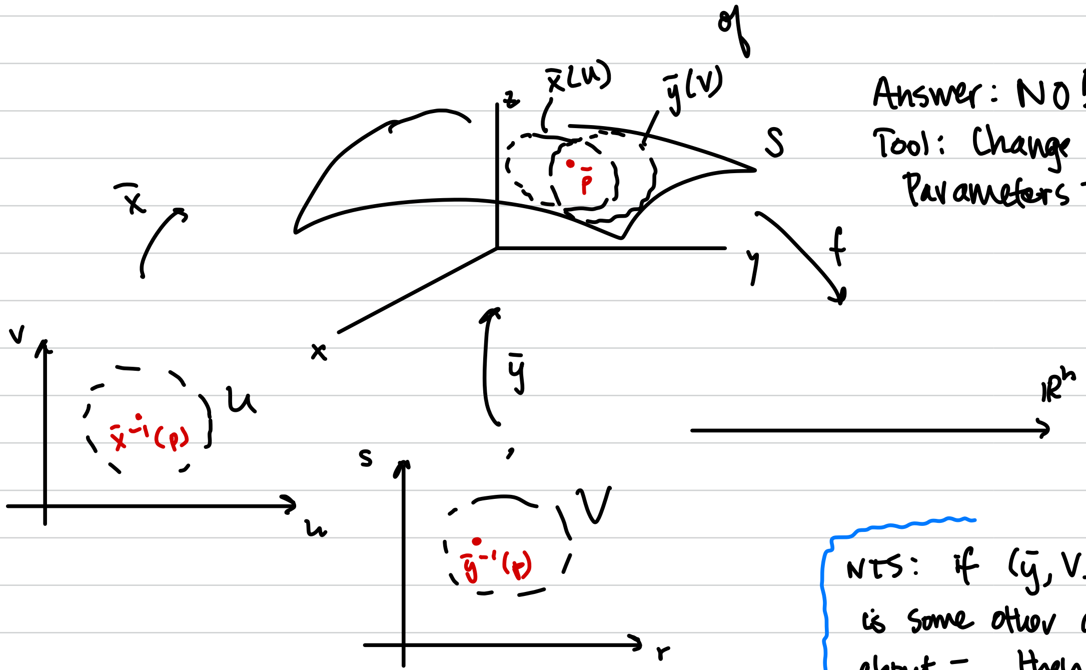
$$f\circ\bar{x}_N(u,v) = f\left(u,v,\sqrt{1-u^2-v^2}\right) = uv\sin\left(\sqrt{1-u^2-v^2}\right)$$

$\frac{\partial}{\partial u}(f\circ x_N)$ and $\frac{\partial}{\partial v}(f\circ\bar{x}_N)$

exist and are cts at $(0,0) = \bar{x}^{-1}(\bar{p}) = \bar{x}^{-1}(0,0,1)$.

Similar for higher order partials. So $f: S \to \mathbb{R}$ is diff'ble at $\bar{p} = (0,0,1)$.

Important Question: Defn of differentiability of $f: S \to \mathbb{R}$ seems to depend on choice of chart. Does it?



Answer: NO!
Tool: Change of Parameters Thm.

NTS: if $(\bar{y}, V)$ is some other chart about $\bar{p}$, then $f \circ \bar{y}: \mathbb{R}^2 \to \mathbb{R}$ is diffble at $\bar{y}^{-1}(\bar{p})$.

$$f \circ \bar{y} = \underbrace{\overbrace{f \circ \bar{x}}^{\mathbb{R}^2 \to \mathbb{R}^n}}_{\text{diffble at } \bar{x}^{-1}(p) \text{ by assumption}} \circ \underbrace{\overbrace{\bar{x}^{-1} \circ \bar{y}}^{\mathbb{R}^2 \to \mathbb{R}^2}}_{\text{diffble by C.o.p. thm.}}$$

By chain rule,
$f \circ \bar{y}$ is composite of diffble functions, so it is diffble at $\bar{y}^{-1}(\bar{p})$ as well. So defn is indep of choice of chart.(!)