Proof of change of parameters theorem depends on Inverse Function Theorem.

Recall:

Thm (Inverse F.T.)

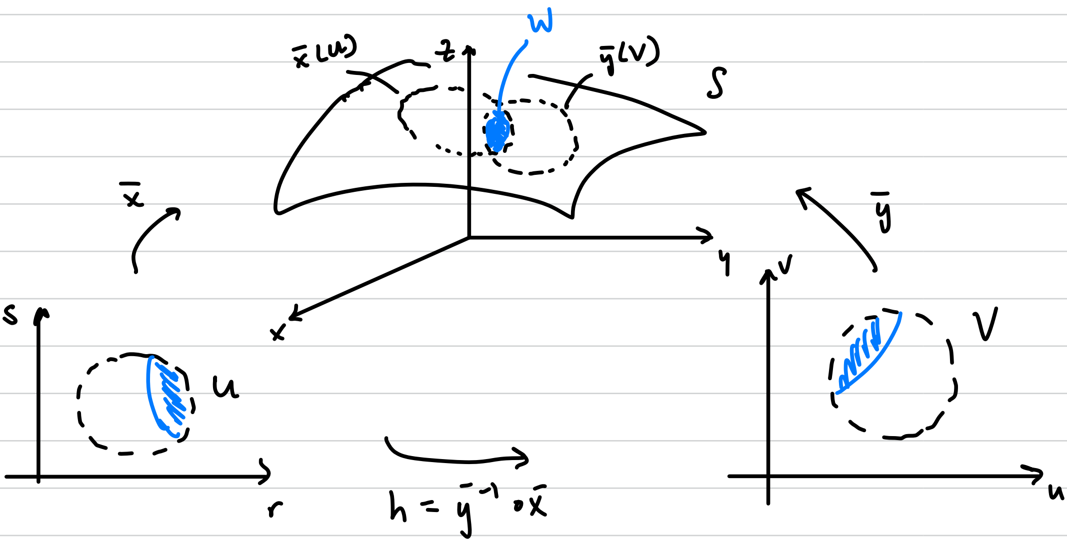
same

Let $F: (U \to \mathbb{R}^n) \to \mathbb{R}^n$ be a diff'ble mapping and suppose that for $\bar{p} \in U$, $dF_{\bar{p}}: \mathbb{R}^n \to \mathbb{R}^n$ has rank n (so $dF_{\bar{p}}$ is 1-1 and onto).

Then there is an open set $V \subset U$ about $\bar{p}$ and an open set $W \subset \mathbb{R}^n$ about $F(\bar{p})$ such that $F: V \to W$ has a diff'ble inverse $F^{-1}: W \to V$.

Idea: dF_p is a linear approx. of F. If $dF_{\bar{p}}$ is 1-1 and onto then near $\bar{p}$, so is F (and F^{-1} is diff'ble).

Proof (change of parameters)



NTS: h a diffeomorphism:
- 1-1, onto
- diff'ble
- h^{-1} diff'ble.

Since $\bar{x}$ and $\bar{y}$ are 1-1 and onto their images,

$h = \bar{y}^{-1} \circ \bar{x}$ is 1-1 and onto

So: NTS: h is diff'ble

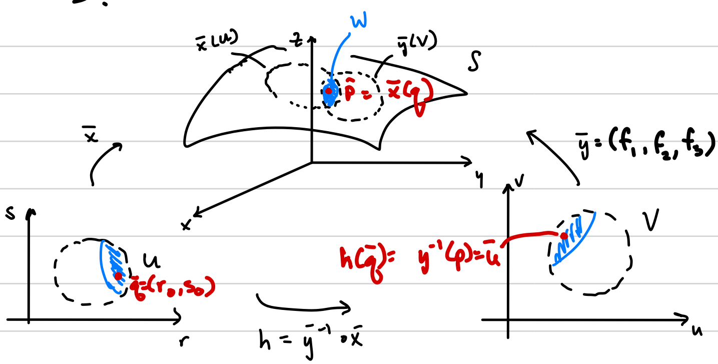
By symmetry, h^{-1} is diff'ble.

Note: we know $\bar{x}: \mathbb{R}^2 \to \mathbb{R}^3$ is diff'ble but we don't know what it means for $y^{-1}|_W$ to be diff'ble (yet!). ↑surface

So can't use chain rule. ⇝ $d(f \circ g)_p = df_{g(p)} \cdot dg_p$

Need to "puff" $\bar{y}^{-1}$ up to a function $\mathbb{R}^3 \to \mathbb{R}^2$.

Sps $\bar{q} = (r_0, s_0) \in \bar{x}^{-1}(W)$.



We know

$\bar{y} = (f_1, f_2, f_3)$

is diff'ble at

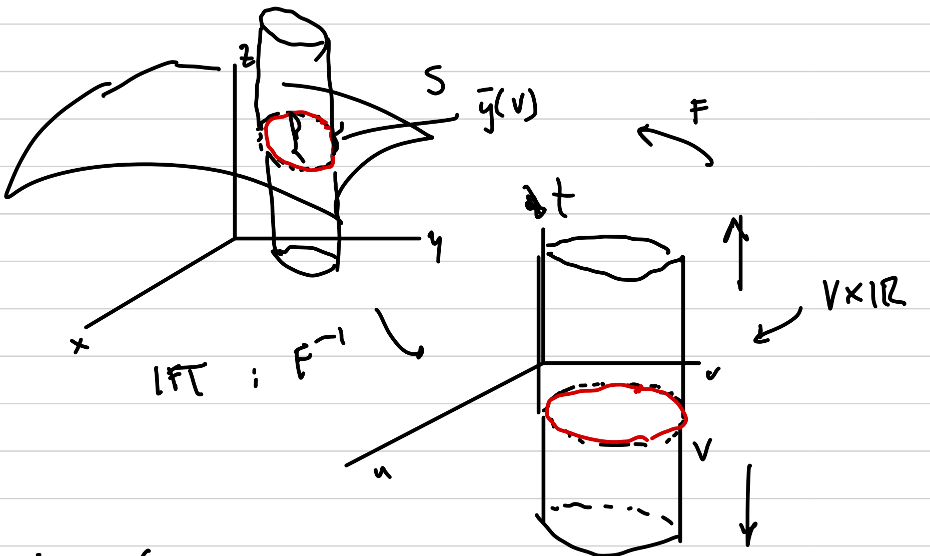
$h(\bar{q}) = \bar{y}^{-1}(\bar{p}) = \bar{u}$

and $d\bar{y}_{\bar{u}}$ has rank 2. → $\begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{bmatrix}$ → $\bar{y}_u \times \bar{y}_v \neq \bar{0}$

W.l.o.g., assume $\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u} \neq 0$ at $\bar{u}$.

Consider $F: \underbrace{V \times \mathbb{R}}_{\subseteq \mathbb{R}^3} \to \mathbb{R}^3$ given by

$$F(u,v,t) = (f_1(u,v), f_2(u,v), f_3(u,v)) + (0,0,t)$$



F is diff'ble (b/c has cts partials all orders) and

$$dF_{(u,v,t)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & 0 \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & 0 \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & 1 \end{bmatrix}$$

$$\det(dF_{(u,v,t)}) = 1 \cdot \left(\frac{\partial f_1}{\partial u} \frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v} \frac{\partial f_2}{\partial u} \right) \neq 0$$

*

$\therefore$ so $dF_{(u,v,t)}$ has rank: 3

open $\qquad\qquad\qquad\qquad$ $\bar{y}(u)$

Thus: by IFT, there is a neighborhood W^+ about $\bar{p} \in \mathbb{R}^3$ on which $F^{-1}: W^+ \to \mathbb{R}^3$ exists and is diff'ble.

Finally, by chain rule: $F^{-1} \circ \bar{x} : \bar{x}^{-1}(W) \to \bar{y}^{-1}(W)$ is diff'ble.

But for $(r_0, s_0) \in \bar{x}^{-1}(W)$,

$$\bar{F}^{-1} \circ \bar{x}\,(r_0, s_0) = (\,h(r_0, s_0),\ 0)$$

Ignoring the last zero (ie composing with projection $\mathbb{R}^3 \to \mathbb{R}^2$ $\quad (u,v,t) \mapsto (u,v)$) gives that $h(r,s)$ is diff'ble at $\bar{q} = (r_0, s_0)$.

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* * * This thm allows us to define what it means for $f: S \to \mathbb{R}^n$ or $f: S_1 \to S_2$ to be diff'ble.