Prop. Let $\bar{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization such that $\bar{p} = \bar{x}(\bar{q})$ where $\bar{q} \in U$. Then

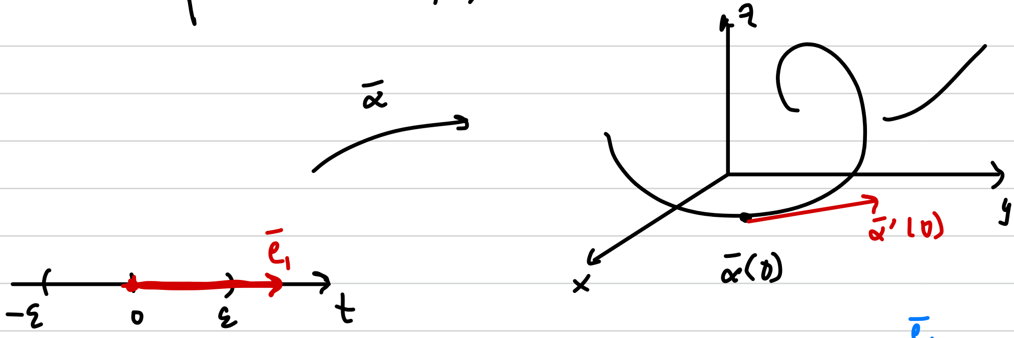
$$d\bar{x}_{\bar{q}}(\mathbb{R}^2) = T_p S$$

proof NTS: ① $T_p S \subset d\bar{x}_q(\mathbb{R}^2)$

② $d\bar{x}_{\bar{q}}(\mathbb{R}^2) \subset T_p S$ "contained in"

$\Rightarrow d\bar{x}_q(\mathbb{R}^2) = T_p S.$

Recall: Sps $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$. Then



$$\bar{\alpha}'(0) = d\bar{\alpha}_0(\bar{e}_1) = \begin{bmatrix} \frac{dx}{dt}(0) \\ \frac{dy}{dt}(0) \\ \frac{dz}{dt}(0) \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$\bar{e}_1$

$*$

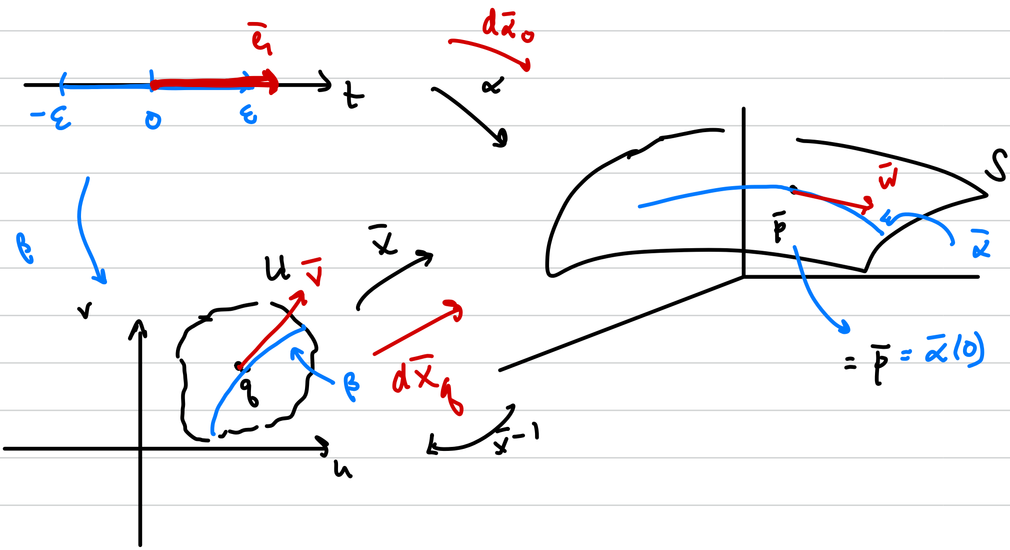
① (Show $T_pS \subset d\bar{x}_q(\mathbb{R}^2)$.)

based at $\bar{q}$

Sps $\bar{w} \in T_pS$. (Need a vector $\bar{v} \in \mathbb{R}^2$ s.t. $\bar{w} = d\bar{x}_q(\bar{v})$)

By defn, since $\bar{w} \in T_pS$, $\bar{w} = \bar{\alpha}'(0)$ where

$\bar{\alpha} : (-\varepsilon, \varepsilon) \to S$ and $\bar{\alpha}(0) = \bar{p}$.



Since $\bar{x}$ is 1-1 and onto its image, use $\bar{x}^{-1}$ to pull $\bar{\alpha}$

back to U:

Let $\bar{\beta} = \bar{x}^{-1} \circ \alpha \ : (-a, a) \to U$ * candidate for

$d\bar{x}_q(\bar{v}) = \bar{w}$.

Then $\bar{\beta}(0) = \bar{q}$ Let $\bar{v} = \bar{\beta}'(0)$ NTS

GOAL: $\bar{w} = d\bar{x}_q(\bar{v})$

Then $\bar{w} = \bar{\alpha}'(0)$

$= (\bar{x} \circ \bar{x}^{-1} \circ \bar{\alpha})'(0)$

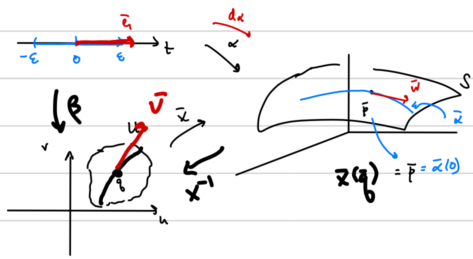
reexpress $= d(\bar{x} \circ \bar{x}^{-1} \circ \bar{\alpha})_0(\bar{e}_1)$ ← unit basis vector for $\mathbb{R}$

chain rule $= d\bar{x}_q \, d(\bar{x}^{-1} \circ \bar{\alpha})_0(\bar{e}_1)$

substitution $= d\bar{x}_q \, d\beta_0(\bar{e}_1)$

reexpress $= d\bar{x}_q(\beta'(0))$

$= d\bar{x}_q(\bar{v}).$



Thus, $\bar{w} \in d\bar{x}_q(\mathbb{R}^2)$, and therefore $T_pS \subset d\bar{x}_q(\mathbb{R}^2)$. ✓

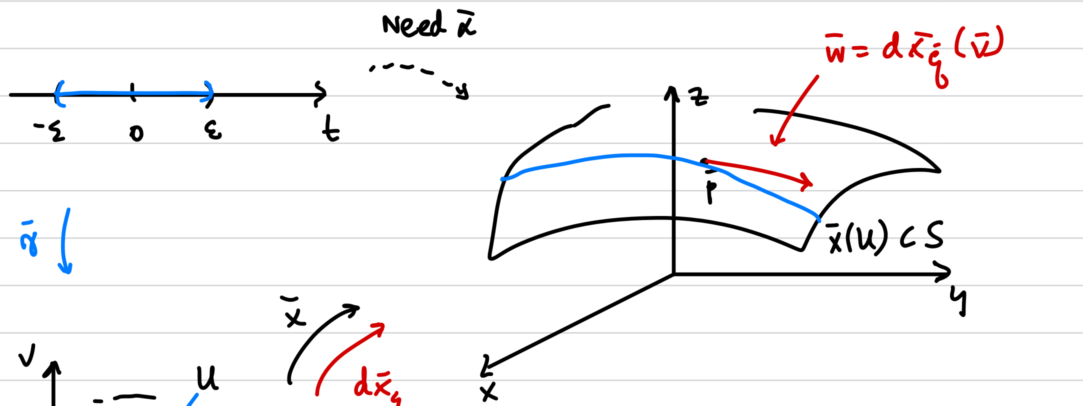
② (show $d\bar{x}_{\bar{q}}(\mathbb{R}^2) \subset T_pS$)

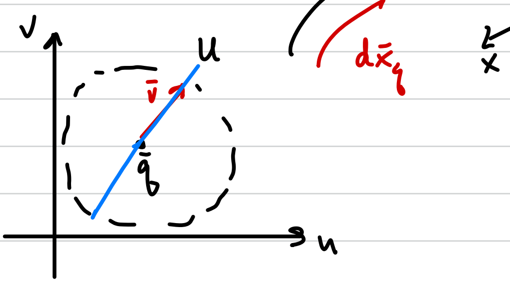
Sps. $\bar{w} \in d\bar{x}_{\bar{q}}(\mathbb{R}^2)$, i.e. $\bar{w} = d\bar{x}_{\bar{q}}(\bar{v})$

for some $\bar{v} \in \mathbb{R}^2$ based at $\bar{q}$.

(Need: a curve $\bar{\alpha} : (-\varepsilon, \varepsilon) \to S$ s.t.

$\bar{\alpha}(0) = \bar{p}$ and $\bar{\alpha}'(0) = \bar{w}$).



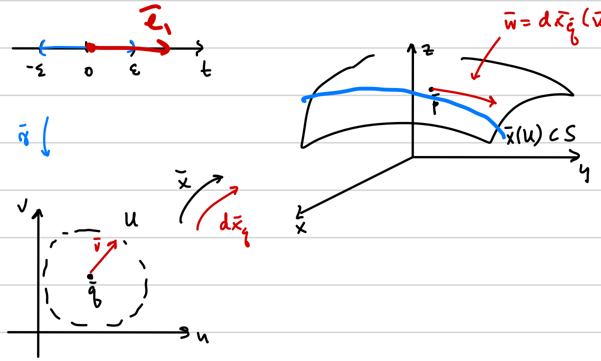


line in U.

Let $\bar{\gamma}(t) = \bar{q} + t\bar{v}$ $\quad t \in (-\varepsilon, \varepsilon)$

Then $\bar{\gamma}(0) = \bar{q}$ and $\bar{\gamma}'(0) = \bar{v}$.

Let $\bar{\alpha} = \bar{x} \circ \bar{\gamma}(t)$ $\quad$ (so: $\bar{\alpha}(t) \in S \ \forall\, t \in (-\varepsilon, \varepsilon)$) ✓



Then $\bar{\alpha}(0) = \bar{x} \circ \bar{\gamma}(0)$

$= \bar{x}(\bar{q}) = \bar{p}$. ✓

and $\bar{\alpha}'(0) = (\bar{x} \circ \bar{\gamma})'(0)$

reexpress $\quad = d(\bar{x} \circ \bar{\gamma})_0(\bar{e}_1)$

chain rule $\quad = d\bar{x}_q \, d\bar{\gamma}_0(\bar{e}_1)$

reexpress $\quad = d\bar{x}_q(\gamma'(0))$

$= d\bar{x}_q(\bar{v})$

$= \bar{w}$. ✓

So $\bar{w}$ is the tangent to a curve in S, i.e. $\bar{w} \in T_pS$.

Thus $d\bar{x}_q(\mathbb{R}^2) \subset T_pS$.

↳ conclusion: $d\bar{x}_q(\mathbb{R}^2) = T_pS$.