Ex. Surfaces defined by equations.

Defn For a diff'ble map $f: \mathbb{R}^n \to \mathbb{R}^m$ with $n \geq m$, we say $\bar{p} \in \mathbb{R}^n$ is a <u>critical pt</u> of f if

df_p is <u>not</u> onto.

i.e. df_p does not have full rank.

The image $f(\bar{p})$ is called a <u>critical value</u>.

Any point $\bar{q}$ in the image of f that is not a critical value is called a <u>regular value</u>.

Ex $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z) = x^2+y^2+z^2 \quad \bar{p} = (x_0, y_0, z_0)$

$$df_{(x_0,y_0,z_0)} = \begin{bmatrix} 2x_0 & 2y_0 & 2z_0 \end{bmatrix}$$

same entries as grad f, also denoted ∇f eval. at (x_0, y_0, z_0)

Since LT df_p maps to $\mathbb{R}$, there are two options:

- image of df_p is all of $\mathbb{R}$ (onto) ← rank = 1

- image is 0 ← rank = 0

For $df_{(x_0,y_0,z_0)} = [2x_0 \quad 2y_0 \quad 2z_0]$.

image is 0 when $2x_0 = 2y_0 = 2z_0 = 0$, ie. $(0,0,0)$

↳ so $(0,0,0)$ is a c.p. with critical value

$$f(0,0,0) = 0^2 + 0^2 + 0^2 = 0$$

critical value.

OTOH: for any $r \neq 0$, $r > 0$, r is an example of a regular value

any pt (x_0, y_0, z_0) s.t. $f(x_0,y_0,z_0) \neq 0$ is <u>not</u> a critical pt.

Thm If $f: \mathbb{R}^3 \to \mathbb{R}$ is a diff'ble map and a is a regular value in the image of f, then

$$\underbrace{f^{-1}(a)}_{\text{a set}} = \{(x,y,z) \mid f(x,y,z) = a\}$$

is a regular surface in $\mathbb{R}^3$ ← domain of f.

Ex. $f(x,y,z) = x^2+y^2+z^2$ $\quad a = 1$

$$f^{-1}(a) \text{ is: } f^{-1}(1) = \{(x,y,z) \mid \underbrace{x^2+y^2+z^2}_{f(x,y,z)} = \underbrace{1}_{a}\}$$

Thm says this is a regular surface.

Remarks:

1. This is an easy way to find regular surfaces
2. No need to find atlas of charts. Guaranteed by proof of thm.
3. The proof depends on the Inverse Function Thm.

Thm (Inverse F.T.)

same

Let $F : (U \to \mathbb{R}^n) \to \mathbb{R}^n$ be a diff'ble mapping and suppose that for $\bar{p} \in U$, $dF_{\bar{p}} : \mathbb{R}^n \to \mathbb{R}^n$ has rank n (so $dF_{\bar{p}}$ is 1-1 and onto).

Then there is an open set $V \subset U$ about $\bar{p}$ and an open set $W \subset \mathbb{R}^n$ about $F(\bar{p})$ such that $F : V \to W$ has a diff'ble inverse $F^{-1} : W \to V$.

Idea: $dF_{\bar{p}}$ is a linear approx. of F. If $dF_{\bar{p}}$ is 1-1 and onto then near $\bar{p}$, so is F (and F^{-1} is diff'ble).