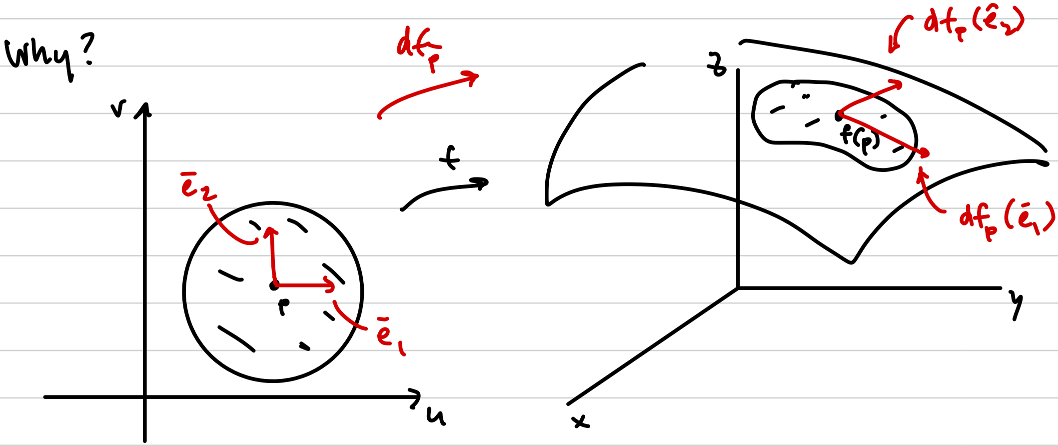
We said std matrix of $f: \mathbb{R}^2 \to \mathbb{R}^3$ at $\bar{p}$ is:

$$df_{\bar{p}} = \begin{bmatrix} \frac{\partial x}{\partial u}(\bar{p}) & \frac{\partial x}{\partial v}(\bar{p}) \\ \frac{\partial y}{\partial u}(\bar{p}) & \frac{\partial y}{\partial v}(\bar{p}) \\ \frac{\partial z}{\partial u}(\bar{p}) & \frac{\partial z}{\partial v}(\bar{p}) \end{bmatrix}$$

Why?



So: std matrix of df_p given by

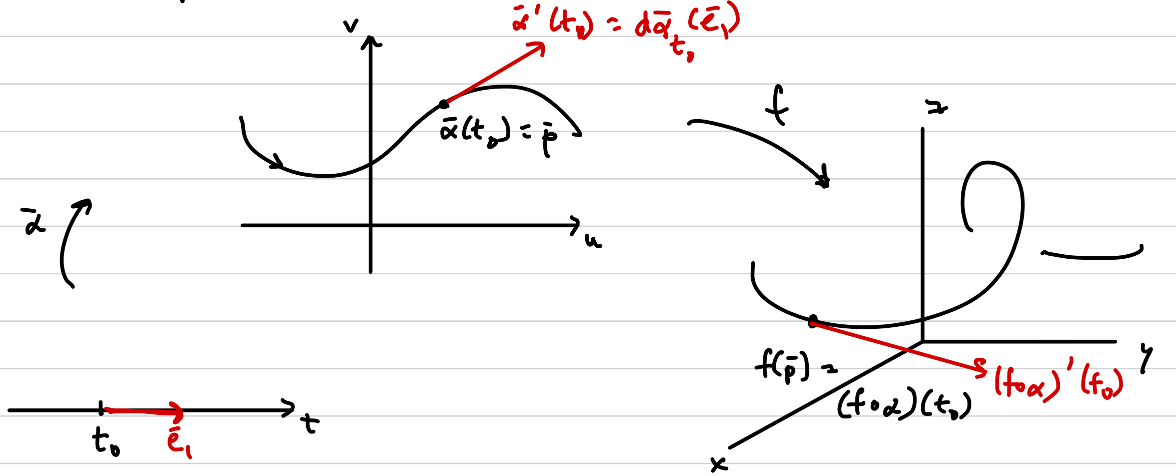
$$\begin{bmatrix} | & | \\ df_p(\bar{e}_1) & df_p(\bar{e}_2) \\ | & | \end{bmatrix}$$

Consider $df_p(\bar{e}_1)$.

Important fact:

Sps $\bar{\alpha} : \mathbb{R} \to \mathbb{R}^2$ a curve s.t. $\bar{\alpha}(t_0) = \bar{p}$.

Sps $f : \mathbb{R}^2 \to \mathbb{R}^3$. Then $f \circ \bar{\alpha}$ a curve in $\mathbb{R}^3$ and

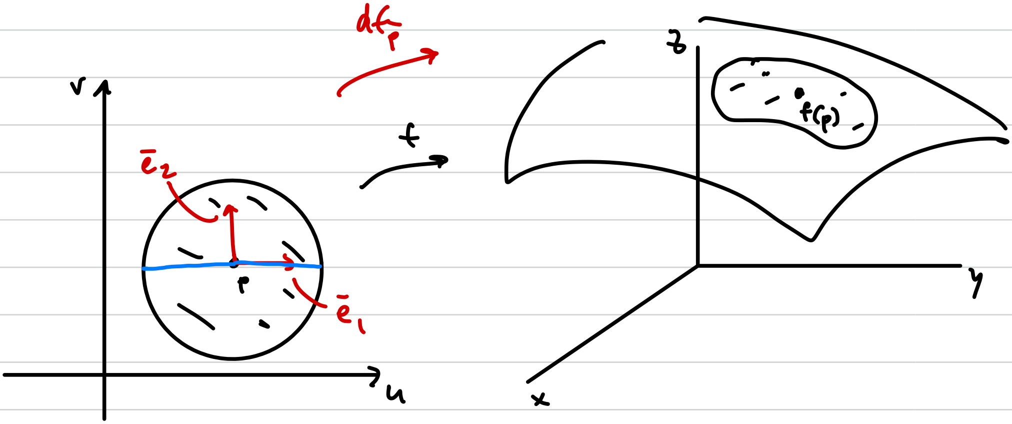


$$\bar{\alpha}'(t_0) = d\bar{\alpha}_{t_0}(\bar{e}_1)$$

And $df_{\bar{p}}(\bar{\alpha}'(t_0)) = df_p(d\alpha_t(\bar{e}_1)) = d(f \circ \bar{\alpha})_{t_0}(\bar{e}_1) = (f \circ \alpha)'(t_0)$.

chain rule

Now, apply this to our situation:



Sps $\bar p = (u_0, v_0)$. Then

$\bar\alpha(u) = (u, v_0)$ has $\bar\alpha(u_0) = \bar p$ and $\bar\alpha'(u_0) = (1,0) = \bar e_1 \in \mathbb{R}^2$.

$f(u,v) = (x(u,v), y(u,v), z(u,v))$

So: $df_p(\bar e_1) = df_p(\bar\alpha'(u_0))$

$= (f \circ \alpha)'(u_0)$

1st column of matrix of $df_{\bar p}$

$= \frac{d}{du}(x(u,v_0), y(u,v_0), z(u,v_0))$

$= \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\right)$

Similarly, second column of $df_{\bar{p}}$ is $\begin{bmatrix} \frac{\partial x}{\partial v}(\bar{p}) \\ \frac{\partial y}{\partial v}(\bar{p}) \\ \frac{\partial z}{\partial v}(\bar{p}). \end{bmatrix}$

Finally, recall: the <u>rank</u> of a matrix is the dimension of the space spanned by the columns.

dimension of image.

With two columns, three possibilities:

Ex.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

image is: dim 0, rank = 0   dim 1, rank = 1   dim 2, rank = 2.