still assuming $\bar{\alpha}(s)$ p.b.a.l.

Now, let's consider further relationships b/w $\bar{t}(s)$, $\bar{n}(s)$, and $\bar{b}(s)$.

Recall: $\{\bar{t}(s), \bar{n}(s), \bar{b}(s)\}$ an orthonormal frame.

Consider $\bar{b}'(s)$:

Since $|\bar{b}(s)| \equiv 1$, we have

$$\bar{b}(s) \cdot \bar{b}(s) = 1 \text{ for all } s.$$

So $$0 = \frac{d}{ds}(\bar{b}(s) \cdot \bar{b}(s)) = 2\bar{b}(s) \cdot \bar{b}'(s).$$

Thus: $\bar{b}'(s) \perp \bar{b}(s)$ ①

Also: $\bar{b}(s) \cdot \bar{t}(s) = 0$ for all s, so

$$0 = \frac{d}{ds}(\bar{b}(s) \cdot \bar{t}(s)) = \bar{b}'(s) \cdot \bar{t}(s) + \bar{b}(s) \cdot \underbrace{\bar{t}'(s)}_{\kappa(s)\bar{n}(s)}$$

0

Thus: $\bar{b}'(s) \perp \bar{t}(s)$ ②

① and ② $\Rightarrow$ $\bar{b}'(s) \parallel \bar{n}(s)$.

"tan"

so $\bar{b}'(s) = \tau(s)\,\bar{n}(s)$ for some function $\tau(s)$.

$\tau(s)$ a scalar function of s.

Defn The function $\tau(s)$ is called the torsion of $\bar{\alpha}$ at s.

$|\tau(s)| = |\bar{b}'(s)|$ = measurement of "lifting".

Note: For a plane curve $\bar{\alpha}(s)$, $\tau(s) = 0$ for all s.

Ex: Torsion of a helix is constant.

Thm (Serret - Frenet formulae)

B/w $\kappa(s)$, $\tau(s)$, $\bar{t}(s)$, $\bar{n}(s)$, and $\bar{b}(s)$, we have the following relations:

$$\bar{t}'(s) = \kappa(s)\bar{n}(s) \qquad Ⓐ$$

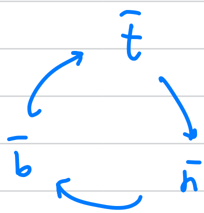
$$\bar{n}'(s) = -\kappa(s)t(s) - \tau(s)\bar{b}(s) \qquad Ⓑ$$

$$\bar{b}'(s) = \tau(s)\bar{n}(s) \qquad Ⓒ$$

proof: Ⓐ, Ⓒ done.

Ⓑ $\bar{n}(s) = \bar{b}(s) \times \bar{t}(s)$

so $\bar{n}'(s) = \underbrace{\bar{b}'(s)}_{\tau(s)\bar{n}(s)} \times \bar{t}(s) + \bar{b}(s) \times \underbrace{\bar{t}'(s)}_{\kappa(s)\bar{n}(s)}$



$$= -\tau(s)\bar{b}(s) - \kappa(s)\bar{t}(s)$$