Space curves: Serret-Frenet frames, curvature, and torsion

Recall: properties of cross product in $\mathbb{R}^3$:

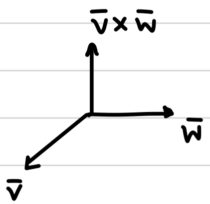
- If $\bar{v}, \bar{w} \in \mathbb{R}^3$ nonzero and $\bar{v} \nparallel \bar{w}$, then

  $\bar{v} \times \bar{w} \perp \bar{v}$ and $\bar{v} \times \bar{w} \perp \bar{w}$.

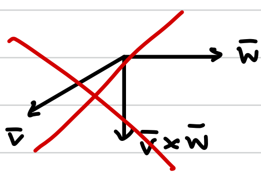
  what if $\bar{v} \parallel \bar{w}$? $\bar{v} \times \bar{w} = \bar{0}$.

- $\bar{v} \times \bar{w} = -\bar{w} \times \bar{v}$ (anticommutative)

- satisfies right-hand rule:

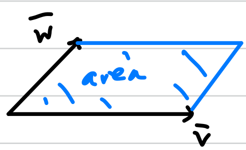


yes



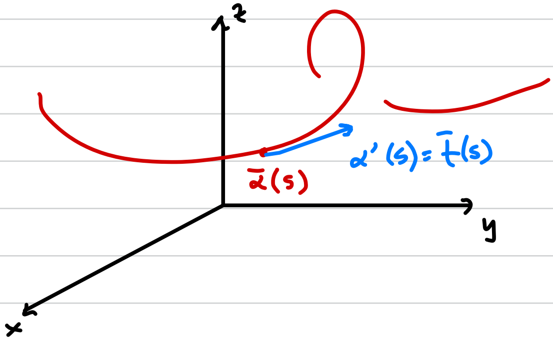
no

- $|\bar{v} \times \bar{w}| =$



Now, sps $\bar{\alpha}: I \to \mathbb{R}^3$ is parametrized by arc length, s.

So: $|\bar{\alpha}'(s)| \equiv 1$ $\equiv$ means equal for all s.



The unit tangent to $\bar{\alpha}$ at s, $\bar{t}(s)$, given by: $\bar{\alpha}'(s)$.

GOAL: An orthonormal basis (a.k.a. a frame) at each pt. along $\bar{\alpha}$.

$\{ \bar{t}(s), \ ? \ , \ ? \}$

will allow us to make measurements about $\bar{\alpha}$.

$\forall$ means "for all"

Consider $\bar{\alpha}''(s)$.

We know $|\bar{\alpha}'(s)| \equiv 1$ so $\bar{\alpha}'(s) \cdot \bar{\alpha}'(s) = 1 \ \forall s$.

But then:

$$0 = \frac{d}{ds}\left(\bar{\alpha}'(s) \cdot \bar{\alpha}'(s)\right)$$

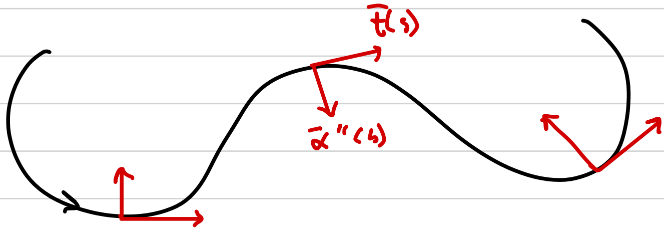
hw

$$= \bar{\alpha}''(s) \cdot \bar{\alpha}'(s) + \bar{\alpha}'(s) \cdot \bar{\alpha}''(s)$$

$$= 2\left(\bar{\alpha}'(s) \cdot \bar{\alpha}''(s)\right)$$

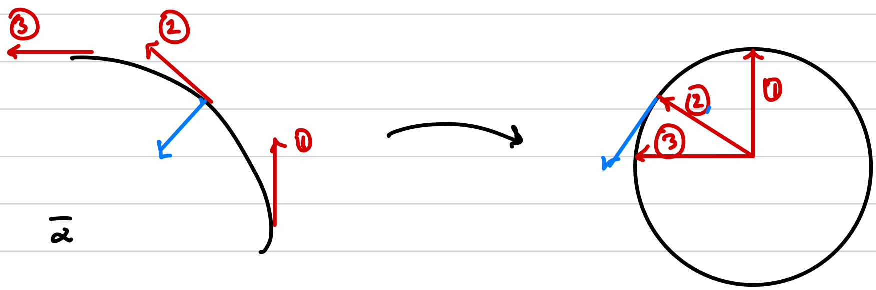
$\bar{t}(s)$

Conclusion: $\bar{\alpha}'(s) \perp \bar{\alpha}''(s)$.



$\bar{\alpha}''(s)$ should measure change of $\bar{\alpha}'(s)$. Length of $\bar{\alpha}'(s) \equiv 1$, so only its direction changes.

So $|\bar{\alpha}''(s_0)|$ measures rate at which $\bar{\alpha}'(s)$ pulls away from tangent line at $\bar{\alpha}(s_0)$.



circle

<u>Defn</u> For a curve $\bar{\alpha} : I \to \mathbb{R}^3$ that is p.b.a.l.,

$\|\bar{\alpha}''(s)\|$, denoted $\kappa(s)$, is the <u>curvature</u>

of $\bar{\alpha}$ at s.

NOTE: $\kappa(s) \geq 0$.

When $\kappa(s) \neq 0$, Let $\bar{n}(s)$ be the unit vector in the direction of $\bar{\alpha}''(s)$. So

$$\bar{\alpha}''(s) = \kappa(s)\,\bar{n}(s).$$

$\bar{n}(s)$ is called the <u>unit normal</u> to $\bar{\alpha}$ at s.

So far: $\{\bar{t}(s), \bar{n}(s), ?\}$

these determine the <u>osculating plane</u> at s.